# The Effects of Competition and Regulation on Error Inequality in Data-Driven Markets

Hadi Elzayn
University of Pennsylvania*
hads@sas.upenn.edu

Benjamin Fish
Microsoft Research
benjamin.fish@microsoft.com

## ABSTRACT

Recent work has documented instances of *unfairness* in deployed machine learning models, and significant researcher effort has been dedicated to creating algorithms that intrinsically consider fairness. In this work, we highlight another source of unfairness: market forces that drive differential investment in the data pipeline for differing groups. We develop a high-level model to study this question. First, we show that our model predicts unfairness in a monopoly setting. Then, we show that under all but the most extreme models, competition does not eliminate this tendency, and may even exacerbate it. Finally, we consider two avenues for regulating a machine-learning driven monopolist - relative error inequality and absolute error-bounds - and quantify the price of fairness (and who pays it). These models imply that mitigating fairness concerns may require *policy*-driven solutions, not only technological ones.

## CCS CONCEPTS

• **Theory of computation → Market equilibria**; *Machine learning theory*; *Sample complexity and generalization bounds*; *Quality of equilibria*; • **Applied computing → Economics**.

## KEYWORDS

learning theory, algorithmic fairness, data markets, game theory, industrial organization, economics

**ACM Reference Format:**
Hadi Elzayn and Benjamin Fish. 2020. The Effects of Competition and Regulation on Error Inequality in Data-Driven Markets. In *Conference on Fairness, Accountability, and Transparency (FAT* '20), January 27–30, 2020, Barcelona, Spain.* ACM, New York, NY, USA, 18 pages. https://doi.org/10.1145/3351095.3372842

## 1 INTRODUCTION

As machine learning has become more integrated into products, markets, and decision-making throughout society, researchers, practitioners, and activists have identified many instances of *unfairness* in predictions or decisions made by machine-learned models (or by humans influenced by said models). A large and developing body of work, which we briefly survey in Section 2, has empirically documented unfairness in practical machine learning settings, identified many theoretical sources and mechanisms of unfairness, and constructed innovative fairness-aware algorithms. Researchers have developed many innovative technical solutions to these problems, yet the issue in practice remains far from solved. This paper highlights a simple and important point: while technical solutions to unfairness are certainly important, mitigating unfairness in practice may require tackling *economic* incentives promoting unfairness.

Most of the existing literature assumes that a fixed dataset, possibly biased, arrives in the hands of a data scientist, and solutions often revolve around clever ways to mitigate this bias. In practice, however, economic incentives may create disparities well before the data scientist enters the picture. Consider, for example, the task of speech recognition: producing accurate models may require a large amount of data, and data from speakers with accented or rarer dialects may be more costly to collect. If the market size of a minority group is small relative to the costs a firm would expend in developing accurate speech recognition software, it is likely that the group will be served with lower quality products.

In this paper, we model the unfairness that arises when data-driven, profit-maximizing firms choose to differentially invest in data collection across groups, creating unequal error rates. In order to focus on this specific source of unfairness, we use a simple framework that elides the many other sources of bias that can seep into the machine learning pipeline. For instance, we assume that firms have unlimited budgets to purchase data at a cost from group-specific data sources of potentially infinite quantity. We also assume that both firms and users benefit from more accurate models, so that incentives are aligned. Furthermore, we assume that firms must build separate models for each group, to avoid unfairness that may come from fitting to the majority.

In order to construct our models, we borrow from the tools of learning theory and microeconomics to build simple, stylized models with crisp predictions of quantifiable unfairness. We assume each profit-maximizing firm faces a known demand curve as a function of the *worst-case* error rates for each group. Standard results from learning theory allow us to model worst-case error rates as a function of the amount of data the firm buys. We investigate three models of demand: linear demand, demand proportional to error rates, and (approximately) rational demand. For the precise description of our models and these assumptions, see Section 3.

We show in Section 4 that a profit-maximizing monopolist will choose to serve minorities (as defined by their market power) with lower quality models. Assuming linear demand, an oft-used benchmark in the economics literature, we quantify the difference in relative model quality between groups as a function of their market size, elasticity, and cost of data.

*This work was completed while the author was an intern at Microsoft Research.


*FAT* '20, January 27–30, 2020, Barcelona, Spain*

ACM ISBN 978-1-4503-6936-7/20/01…$15.00
https://doi.org/10.1145/3351095.3372842

We then consider two classical remedies to the ills of monopolies: competition and regulation. Under two natural models of competition – multilinear demand (Section 5.1) and proportional demand (Section 5.2) – introducing competition does not mitigate inequality, and proportional demand even exacerbates it. Only a model in which all consumers choose the firm with (even infinitesimally) smaller error until firms reach sufficient accuracy suggests that competition will mitigate inequality (Section 5.3); to do so, however, this model assumes a stringent notion of rationality that may not be reflective of consumer behavior in the real world.

Given that our most plausible models suggest that competition does improve the situation, we ask whether regulation could be used to mitigate error inequality by design. In particular, in Section 6 we examine two simple kinds of constraints: a 'relative equality' constraint where error rates across groups must be multiplicatively close to each other, and an 'absolute equality' constraint where error rates across all groups must be sufficiently small, but may be far apart from each other. We then formally quantify the costs to profits (and when relevant, to the majority group's error rate) as a function of the threshold chosen. Finally, we conclude with takeaways, limitations, and directions for future work in Section 7.

## 2 RELATED WORK

Motivation for our work comes from the many documented instances of disparity in learned model performance between groups. The existing literature has demonstrated troubling disparities in a number of domains, including incentive-aligned domains (where both the firms and users receive benefit from more accurate models) that are the focus of this work. Wilson et al. [34] studies the performance of state-of-the-art object recognition systems, intended for applications like autonomous vehicles, and find that systems fail to recognize darker-skinned persons at much higher rates than lighter ones. Sweeney shows that search engine queries of black-associated names generated about four times the likelihood of ads for arrest records [30]. Blodgett and O'Connor show that on both complicated tasks like parsing and simple tasks like language identification, texts from speakers of African American English see vastly higher error rates [8]. Buolamwini and Gebru show that commercial facial recognition software systems misclassify race and gender among dark-skinned females at orders of magnitude higher rates than light-skinned males [9]. Mehrotra et al. [26] and Ekstrand et al. [13] identify differing satisfaction levels by age and gender in recommendation systems. The list goes on.

Researchers have engaged in many empirical and theoretical investigations to understand why these instances of unfairness occur, with the hope of developing solutions to mitigate them. Much of this work focuses on the learning algorithm itself as the source of unfairness, and attempts to incorporate fairness notions into the algorithm [21]; see e.g. Verma and Rubin [33] for a survey of fairness definitions. Training data has also been identified as source of unfairness; for example, Chen et al. identify sample size differences as a crucial source of unfairness, and decomposes induced unfairness into bias, variance, and noise [10]. Various feedback loops stemming from historical bias have also been identified as sources of unfairness [14, 15, 24]. There are a few others, including selection bias [20], using the wrong metric [27], or using a single model across multiple underlying data generating processes [22]. However, to the best of our knowledge, market forces in data investment have seen little attention as a source of unfairness. See the survey of Cowgill and Tucker [11] for an in-depth survey of perspectives on the sources of unfairness from computer science and economics.

Our models are built on insights from two extensive, and historically separate, literatures: the formalization of learning from data embodied in computational and statistical learning theory, and models of strategic interactions from the theory of industrial organization (see e.g. Tirole [31]). From learning theory, we apply fundamental bounds on sample complexity derived from the *Probably Approximately Correct* (PAC) framework (see e.g. Kearns and Vazirani [23]) to relate firms' costs to worst-case error rates; from industrial organization, we modify widely used models of demand (such as linear demand, multilinear models of imperfect substitutes [6], the Tullock contest [32], and Bertrand competition [29]) to link firms' choices to consumer behavior.

Recently, these two fields have drawn closer, as both computer scientists and economists have begun to model markets for information and data. For example, Aridor et al. [1] and Mansour et al. [25] consider the exploration-exploitation tradeoffs faced by firms competing to win users in a bandits setting, while Ben-Porat and Tennenholtz formalize competition in the *prediction space* that can lead to models very different than those produced by empirical risk minimization algorithms [3, 4]. To the best of our knowledge, however, this is the first work to apply learning theory and industrial organization to explore differing incentives in the context of fairness. The work of Dong et al. [12] is the closest in form to ours, and uses a similar high-level abstraction of learning theory, as well as a proportional-error split in market share, but primarily explores questions of market concentration.

## 3 CONSUMER BEHAVIOR AND LEARNING THEORY

We begin by describing our framework at a high level. In our models, firms use data to create a classifier (or other machine learning model) that is then used to serve consumers. Consumers are split into non-overlapping groups, and choose a firm based on how well the firm's model is performing for their group. Firms receive revenue based on how many consumers they attract, but must pay for the amount of the data they buy. The more data, the better their model. The firms aim to maximize their profits. In the case where there are multiple firms, the goal of each firm is to maximize their profit at equilibrium, as other firm's choices affect the number of consumers that they get, and hence their choices. Here, the firm's only (strategically relevant) choice is how much data to buy.

We start with the monopoly case, where there is only one firm. The firm chooses a number of data points $M_g$ to buy for each group, where we write $M$ for the vector of these choices; we write $\varepsilon_g(M_g)$ for the worst-case error the firm can guarantee for group $g$, and assume this error is known to consumers. The groups then respond by entering the market according to a demand function $D_g(\varepsilon_g)$, where $D_g(\varepsilon_g)$ represents the proportion of $g$ that uses the firm's model. Each group has $\mu_g$ total people, so the firm's revenue is

$\sum_{g \in \mathcal{G}} \mu_g D_g(\varepsilon_g(M_g)))$. The firm also pays for the data, represented by a cost function $C(M)$.

We will discuss our choices for $\varepsilon$, $D$, and $C$ in Sections 3.1 and 3.2. But for now, the firm's profit is just the revenue the firm makes minus the cost it spends to acquire that data, leading to the following optimization problem:

Definition 3.1 (The Monopolist's Problem). *The firm chooses $M$ to maximize its profit $\pi(M)$, i.e.*

$$\max_M \pi(M) = \max_M \sum_{g \in \mathcal{G}} \mu_g D_g(\varepsilon_g(M_g)) - C(M).$$

Because we will assume in Section 3.1 that $\varepsilon_g$ is a deterministic function of $M_g$, we can also rewrite this optimization problem as

$$\max_\varepsilon \pi(\varepsilon) = \max_\varepsilon \sum_g \mu_g D_g(\varepsilon_g) - C(\varepsilon),$$

where $\varepsilon$ is the vector of $\varepsilon_g$. We define $\pi(\varepsilon) = \sum_g \mu_g D_g(\varepsilon_g) - C(\varepsilon)$ as the total profit the monopolist makes. We will have an additively separable cost function in $g$, i.e. $C(\varepsilon) = \sum_{g \in \mathcal{G}} C_g(\varepsilon_g)$, which will allow us to also refer to the per-group profit: $\pi_g(\varepsilon_g) = \sum_{g \in \mathcal{G}} \mu_g D_g(\varepsilon_g) - C_g(\varepsilon)$.

On the other hand, when there are multiple firms $\mathcal{F}$, maximizing profit is not longer just an optimization problem, because each firm's optimal choice will depend on its opponent's choice. So instead, we search for a Nash equilibrium, which is the workhorse solution concept in classical game theory. Under such a Nash equilibrium, each firm plays their best response, given all the choices of the other firms. For a more thorough background, see [17].

Extending our notation, we have the same components as in the monopolist case, except now we write $M_{gi}$ for the number of data points the $i$th firm buys for group $g$, $\varepsilon_{gi}$ is the error rate of the $i$th firm on group $g$, and $D_{gi}(\varepsilon_g) = D_{gi}(\varepsilon_{gi}, \varepsilon_{g,-i})$ is the demand for the $i$th firm from group $g$, given the vector $\varepsilon_g = (\varepsilon_{gi})_{i \in \mathcal{F}}$ of error rates.

Definition 3.2 (The Competitor's Problem). *Firms simultaneously announce their choices, resulting in a matrix $M = (M_{gi})$ of data points purchased. Each group in the market responds according to $\varepsilon_g(M_g)$.*

*Then a (pure) equilibrium under profit-maximizing firms is a set of vectors $M_i^*$ chosen with a best response:*

*For all $i$,*

$$M_i^* = \operatorname*{argmax}_{M_i} \pi_i(M_i, M_{-i}^*)$$

*where for any $M$,*

$$\pi_i(M_i, M_{-i}) = \sum_g \mu_g D_{gi}(\varepsilon_{gi}(M_{gi}), \varepsilon_{g,-i}(M_{g,-i})) - C(M_i).$$

We will only consider pure strategy Nash equilibria in this work.

Again, we can write an equivalent definition of the competitor's problem in terms of error:

$$\max_{\varepsilon_i} \pi_i(\varepsilon_i, \varepsilon_{-i}^*) = \max_{\varepsilon_i} \sum_{g \in \mathcal{G}} \mu_g D_{gi}(\varepsilon_{gi}, \varepsilon_{g,-i}^*) - C(\varepsilon_i),$$

where $\varepsilon_i^*$ is the vector of error rates given by the associated equilibrium choice $M_i^*$. We also use $\pi_{gi}(\varepsilon_{gi}, \varepsilon_{g,-i}^*) = \mu_g D_{gi}(\varepsilon_{gi}, \varepsilon_{g,-i}^*) - C_g(\varepsilon_i)$ to refer to the profit $i$ makes on group $g$.

Note that a firm $i$ only enters a market in the first place if $\pi_i(\varepsilon^*) > 0$. In this work, we do not consider the case when $\pi_i(\varepsilon^*) \leq 0$, as our goal is to show that even when firms *do* enter the market for each group, market forces may *still* create a disparity between groups.

Finally, in Section 6, we discuss imposing regulation on a monopolist to ensure some kind of 'fairness' across groups. We consider two different kinds of constraints a regulator could impose on a firm. The first is what we refer to as *relative error equality*, which roughly corresponds to group fairness in binary classification [5] For all $g, g' \in \mathcal{G}$, we require

$$\frac{\varepsilon_g}{\varepsilon_{g'}} \leq (1 + \chi),$$

for parameter $\chi \geq 0$. On the other hand, we could ask for an *absolute error guarantee*, requiring that the error rates for both firms are low, regardless of how close to each other they are: For all $g \in \mathcal{G}$, we require instead

$$\varepsilon_g \leq \chi.$$

This roughly corresponds with maximin notions of fairness, e.g. [5, 7, 16].

We investigate what happens when a monopolist satisfies one of these two constraints. Because error is the relevant quantity from the regulator's perspective, and error and data investment are so tightly linked, we write the regulated monopolist's problem in terms of the choice of error:

Definition 3.3 (Regulated Monopolist's Problem). *The firm chooses $M$ to maximize its profit $\pi(M)$ subject to a constraint:*

$$\max_\varepsilon \pi(\varepsilon) = \max_\varepsilon \sum_g \mu_g D_g(\varepsilon_g) - C(\varepsilon) \text{ s.t. } f_r(\varepsilon) \leq 0 \; \forall r \in R,$$

*where either $R = \mathcal{G} \times \mathcal{G}$ and $f_{g,g'}(\varepsilon) = \varepsilon_g - (1+\chi)\varepsilon_{g'}$, or $R = \mathcal{G}$ and $f_g(\varepsilon) = \varepsilon_g - \chi$.*

Just as is the case in binary classification, where different settings may call for different notions of fairness, which version of fairness regulator should impose will depend on the context and the ethical assumptions she maintains.

## 3.1 Data, Costs, and Learning Theory

A key component to our model is how choices in data investment drive error rates. We assume that firms build a model to provide a product to consumers, and that this model is learned from data. The firms have access to independent and identically distributed data from fixed data sources that reflect the same distribution that consumers care about.

In the PAC model of learning [23], there is a class of hypotheses $H$, and each hypothesis $h \in H$ has an associated risk $R(h)$, typically representing the error rate of $h$. For example, in binary classification, $R(h) = E_{x,y \sim \mathcal{D}}[h(x) \neq y]$, though our model will be applicable to other settings as well. With only access to data drawn from $\mathcal{D}$, rather than $\mathcal{D}$ itself, the learner cannot guarantee its risk, but *can* achieve high probability upper bounds on its risk. In the agnostic PAC setting, there is a learning algorithm that upon seeing a sample of size $M$, except with probability $\delta$, returns a hypothesis $h$ such that

$$R(h) - \min_{h' \in H} R(h') \leq K \sqrt{\frac{d_H + \log(1/\delta)}{M}},$$

where $\min_{h' \in H} R(h')$ is the Bayes error, $d_H$ is the VC dimension of $H$, and $K$ is a universal constant. (See [28] for more on PAC learning, VC dimension, and the various kinds of PAC learning.)

To model the fact that getting appropriate data can have group-dependent sources and thus costs, we assume data for each group is drawn separately from distributions $\mathcal{D}_g$. The firms choose $M_g$, the number of data points to draw, and will use a learning algorithm with a PAC guarantee for each data set and give to a consumer of group $g$ the output of the corresponding hypothesis.

Achieving such bounds would not be useful to the firm unless consumers make decisions based on these bounds. Here, we assume that the consumers have no more access than the firms: they do not have access to the distribution, so they cannot make decisions based on the true group-level error rates. Given this, we assume that the consumers use firms' bound on the excess error $R(h) - \min_{h' \in H} R(h')$, which we refer to as the *worst-case excess error* rate. Of course, in reality, consumer decisions are not necessarily based on the worst-case error rate. However, given that consumers often do in practice have to make choices using relatively little information about firms, and have trouble predicting how well exactly a firm will treat them, we believe this is a natural place to start. In particular, bounds on the the excess error rates represent the minimal amount of information consumers need to make informed decisions.

Thus, we set

$$\varepsilon_{gi}(M_{gi}) = \frac{\gamma_g}{M_{gi}^{1/q}},$$

for constants $\gamma_g > 0$ and $q > 0$. For example, in agnostic PAC learning, $q = 2$ and $\gamma_g = \sqrt{K(d_H + \log(1/\delta))}$. Note that we are assuming $\delta$ is fixed ahead of time, but we allow in general for $\gamma_g$ to be group-dependent. Agnostic PAC learning is far from the only type of learning to have this form; the realizable PAC setting, the multi-class setting, and many regression settings all have this form [28].

This set-up does ignore the possibility of transfer learning, i.e. using the data from $D_g$ to help with learning for another group $g'$. We avoid this scenario so as to concentrate on the 'unfairness' generated via the market incentives instead of the unfairness generated, for example, when an assumption about the similarity between $D_g$ and $D_{g'}$ fails to hold.

The choice of $M_g$ determines not only the worst-case error rates, but the cost to the firm of generating that data, either by collecting it in the wild or buying it from another source. As mentioned above, we permit the costs to be group-dependent. We assume the cost is additively separable and linear in $M_g$:

$$C_{gi}(M_{gi}) = \phi_{gi} + c_{gi} M_{gi} \text{ and } C_i(M_i) = \sum_{g \in \mathcal{G}} C_{gi}(M_{gi}),$$

for constants $\phi_{gi}, c_{gi}$, where $\phi_{gi}$ represents the fixed cost of entering the market.

Since we can rewrite $M_{gi} = (\gamma_g / \varepsilon_{gi})^q$, this model is equivalent to first choosing a worst-case error rate $\varepsilon_{gi}$ and then paying a cost

$$C_{gi}(\varepsilon_{gi}) = \phi_{gi} + \frac{\gamma_g}{\varepsilon_{gi}^q}$$

for each group $g$, where $\gamma_g$ is redefined to minimize the number of constants we employ.

So now

$$M_{gi} = \frac{\gamma_g}{c_{gi} \varepsilon_{gi}^q}.$$

This version is the one we will use for the remainder of the paper. Note that the cost function is convex, which means that whenever the demand is concave, so is the profit function.

## 3.2 Models of Consumer Choice

The firm's revenue is driven by how *demand* for its product reacts to its choice of worst-case error guarantees. We consider several models of this demand, each inspired by well-studied models in microeconomics. While firms are primarily concerned only with aggregate changes in demand, rather than the decisions of individual consumers, each of our models can be founded on natural models of individual consumer behavior, and we provide such models in several cases.

In the monopoly case, we use a simple model of *linear* demand; while an idealization, linear demand is often used even in econometric estimation (see e.g. [18]). In the competitive case, there are a variety of natural demand models, each embedding different assumptions about how consumers choose between firms and how stringently they react to differing error levels. We study three models along a spectrum of rationality: a *multilinear* demand, generalizing the monopoly case; a parameterized *proportional* demand; and an *approximately* rational demand, where consumers exclusively use the firm with lowest error (up to some tolerance).

We give the details of these models of demand in each appropriate subsection in Sections 4 and 5. Under each model, there are parameter regimes where firms choose not to invest in data collection for some groups at all ; while this may reflect some real-world scenarios, the purpose is of this paper is to highlight economic incentives that create inequality *even aside from such extreme scenarios.* As such, we will focus on *interior* optima or equilibria. In an interior optimum, the monopolist must make positive profit (so that it enters the market) and choose error rates strictly smaller than 1 for each group (so that it is investing in data collection for each group). Similarly, interior equilibria require that profits for both firms must be positive and each error rate strictly smaller than 1. Our theorems statements will highlight this focus.

# 4 MONOPOLY

We start with the case where there is one firm in the market and demand is linear:

Definition 4.1 (Linear Demand). *A* linear demand *function for each group g is given by:*

$$D_g(\varepsilon_g) = \alpha_g - \beta_g \varepsilon_g,$$

*where* $0 < \beta \leq \alpha \leq 1$.

A linear demand curve can arise from a simple model of consumer behavior: suppose utility-maximizing consumers consider whether or not to use the product, and only use it if it is above some threshold (equivalent to being better than some 'outside option'). If these thresholds are uniformly distributed over some interval, then demand will be linearly decreasing over an interval. Strictly speaking, this is a *piecewise* linear demand, but this does not greatly affect optimal behavior of the firm - it merely means that it will never choose outside the linearly decreasing range unless they are choosing not to invest in providing quality products at all. For simplicity of our theorem statements, we will assume that parameters are such that the firm's achievable errors are a subset of the linear portion of the demand curve, here $0 < \beta \leq \alpha \leq 1$, but in Appendix A, we generalize to arbitrarily large $\alpha, \beta$ to ensure that our results still qualitatively hold.

Our main result here is the following:

Theorem 4.2 (Monopoly Inequality). *Suppose a monopolist with learning rate q faces linear demand. Then in any interior optimum, for every pair of groups g and g′, the error inequality is given by:*

$$\frac{\varepsilon_g^*}{\varepsilon_{g'}^*} = \left(\frac{\mu_{g'}\beta_{g'}\gamma_g}{\mu_g\beta_g\gamma_{g'}}\right)^{\frac{1}{q+1}}.$$

Again, we focus on an interior optimum. Three factors affect the error gap between the minority and the majority: the size of the minority as a share of the total market; the marginal cost of gathering data on the minority vs. on the majority; and the elasticities of the populations with respect to the error. It is also worth noting that the fundamental nature of the learning problem, via the learning rate $q$, affects the magnitude of error inequality.

Theorem 4.2 is a consequence of the following lemma:

Lemma 4.3. *Suppose a monopolist with learning rate q faces linear demand. Then in any interior optimum, error levels set by the monopolist are given by:*

$$\varepsilon_g^* = \left(\frac{q\gamma_g}{\mu_g\beta_g}\right)^{\frac{1}{q+1}}.$$

Proof. Recall that

$$\pi(\varepsilon) = \sum_{g\in\mathcal{G}} \mu_g\left(\alpha_g - \beta_g\varepsilon_g\right) - \sum_{g\in\mathcal{G}}\left(\phi_g + \frac{\gamma_g}{\varepsilon_g^q}\right).$$

Now, we notice that this profit function is separable into the sum of profits from each market. Differentiating with respect to $\varepsilon_g$ separately and setting to zero,we arrive at the first-order conditions:

$$\frac{\partial\pi}{\partial\varepsilon_g} = -\mu_g\beta_g + \frac{q\gamma_g}{\varepsilon_g^{q+1}} = 0.$$

Solving this equation yields $\varepsilon_g^* = \left(\frac{q\gamma_g}{\mu_g\beta_g}\right)^{\frac{1}{q+1}}$. This is indeed a maximum because profit is concave, so the only alternative is an exterior optimum. □

Notice that if, for all $g$, $\pi_g\left(\varepsilon_g^*\right) > 0$, $\pi_g\left(\varepsilon_g^*\right) > \pi_g(1)$, and $\varepsilon_g^* < 1$, then the interior optimum exists and is unique.

# 5 COMPETITION

In this section, we show that under most reasonable models of competition, the introduction of competition does not mitigate error-inequality compared to the monopoly equilibrium, and may, in fact, increase it. Only under Bertrand-like competition, which assumes consumers are strictly rational, is inequality significantly mitigated. In particular, we show that under both the Tullock and the multi-linear models of demand, the inequality between groups as measured by the error rates does not improve relative to the monopoly case. In the case of the Tullock model, as a function of the relative size of the groups, inequality is actually worse.

## 5.1 Multilinear Demand

Next, we consider a simple generalization of linear demand to the two-firm case. This model can be interpreted as a model of competition in identical products with differing quality levels as in [2], but can also be interpreted (as well as microfounded, and used to estimate structural parameters, as in [6]) as markets for imperfect substitutes.

Definition 5.1 (Multilinear Demand). *The* multilinear linear demand *function is, for firms i and j, and for each group g,*

$$D_{gi}(\varepsilon_{gi}, \varepsilon_{gj}) = \alpha_{gi} - \beta_g\varepsilon_{gi} + \lambda_g\varepsilon_{gj},$$

*where* $0 < \lambda_g < \beta_g \leq \alpha_{gi}$ *and* $\alpha_{gi} + \lambda_g \leq 1$.

We require $\beta_g > \lambda_g$ so that demand reacts more strongly to a firm's own error rates than its opponents – this ensures that if both firms increase error, total demand decreases. The other conditions on the parameters are to ensure that the demand is truly (multi)-linear, as opposed to piece-wise linear.

Note it is not the case that all consumers choose the firm with lower error, as one might expect if the products of the firms were perfect substitutes. Instead, users switch between firms depending on their error rates, and even if firms achieved perfect accuracy, the split of the total market might not be even, as captured by differing $\alpha_{gi}$. This could represent brand loyalty, for example, or perhaps that firms' products are not perfectly identical.

Our main result for this case states that the gap between error rates is the same as in the monopoly case:

Theorem 5.2. *Suppose that two firms with learning rate q compete under multilinear demand. Then in any interior equilibrium, for every pair of groups g and g′, error inequality is given by*

$$\frac{\varepsilon_{gi}^*}{\varepsilon_{g'i}^*} = \left(\frac{\mu_{g'}\beta_{g'}\gamma_{gi}}{\mu_g\beta_g\gamma_{g'i}}\right)^{\frac{1}{q+1}}.$$

Theorem 5.2 is a consequence of the fact that the firm's optimal choice does not depend on its opponent's error rates; that is they have a dominant strategy. This is formalized by the following lemma, which is enough to prove Theorem 5.2:

Lemma 5.3. *Suppose that two firms with learning rate q compete under multilinear demand. Then in any interior equilibrium, error levels are given by*

$$\varepsilon_{gi}^* = \left(\frac{q\gamma_{gi}}{\mu_g\beta_g}\right)^{\frac{1}{q+1}}.$$

Proof. This proof will be very similar to that of Lemma 4.3, only now, the behavior of the other firm will affect profit. Recall

$$\pi_i(\varepsilon_{gi}, \varepsilon_{gj}) = \sum_{g\in\mathcal{G}} \mu_g(\alpha_{gi} + -\beta_g\varepsilon_{gi} + \lambda_g\varepsilon_{gj}) - \sum_{g\in\mathcal{G}} \frac{\gamma_{gi}}{\varepsilon_{gi}^q}.$$

We can see that even though the behavior of the other firm will affect profit, firm $i$ still has a dominant strategy. This is because the first-order conditions do not depend on the other firm:

$$\frac{\partial \pi_i}{\partial \varepsilon_{gi}} = -\mu_g \beta_g + \frac{q\gamma_{gi}}{\varepsilon_{gi}^{q+1}}.$$

This is the same first order condition as in the monopolist's case, with the same implication: $\varepsilon^*_{gi} = \left(\frac{q\gamma_{gi}}{\mu_g \beta_g}\right)^{\frac{1}{q+1}}$.

□

Similar to the case of the monoplist's, notice that if, for all $g$, $i$, $\pi_{gi}\left(\varepsilon^*_{gi}, \varepsilon^*_{gj}\right) > 0$, $\pi_{gi}\left(\varepsilon^*_{gi}, \varepsilon^*_{gj}\right) > \pi_{gi}(1, \varepsilon^*_{gj})$, and $\varepsilon^*_{gi} < 1$, then an interior equilibrium exists.

## 5.2 Proportional Demand

In this section, we consider a model inspired by [12], and thus, indirectly, by the Tullock contest [32]. In particular, firms split the market proportionally to the other firms' error:

Definition 5.4 (Proportional Demand). *In a multi-firm market, we say that demand is* proportionally split *with* competition exponent $\rho$ *if*

$$D_{gi}(\varepsilon) = 1 - \frac{\varepsilon_{gi}^{\rho}}{\sum_j \varepsilon_{gj}^{\rho}} = \frac{\sum_{j \neq i} \varepsilon_{gj}^{\rho}}{\sum_j \varepsilon_{gj}^{\rho}}.$$

*Here, we focus on the two-firm case, in which case we can write*

$$D_{gi}(\varepsilon_{gi}, \varepsilon_{gj}) = 1 - \frac{\varepsilon_{gi}^{\rho}}{\varepsilon_{gi}^{\rho} + \varepsilon_{gj}^{\rho}} = \frac{\varepsilon_{gj}^{\rho}}{\varepsilon_{gi}^{\rho} + \varepsilon_{gj}^{\rho}}.$$

Now we can write our inequality theorem:

Theorem 5.5 (Inequality Under Proportional Demand). *Suppose two firms with learning rate q compete under proportional demand. Then in any interior equilibrium error inequality is given by*

$$\frac{\varepsilon^*_{gi}}{\varepsilon^*_{g'i}} = \left(\frac{\rho_{g'}\mu_{g'}}{\rho_g \mu_g}\right)^{\frac{1}{q}} \cdot \frac{f(\gamma_{gi}, \gamma_{gj}, q)}{f(\gamma_{g'i}, \gamma_{g'j}, q)},$$

*where*

$$f(\gamma_{gi}, \gamma_{gj}, q) = \frac{(\gamma_{gi}^q + \gamma_{gj}^q)^{\frac{2}{q}}}{\gamma_{gi}^{1-\frac{1}{q}} \gamma_{gj}^q}.$$

Recall that in the monopoly case, the exponent was $1/(q+1)$ instead of $1/q$, meaning that introducing competition under this model has actually *exacerbated* the effect of minority status on inequality. Note also that the relative inequality between two groups based on the results from a particular firm depends not only on that firm's cost structure for the two groups, but also on the opposing firm's cost structure for the two groups.

Proving Theorem 5.5 requires finding the equilibrium:

Lemma 5.6. *Suppose two firms with learning rate q face proportional demand with competition exponent $\rho$. In any interior equilibrium, error rates are given by:*

$$\varepsilon^*_{gi} = \left(\frac{q\gamma_{gi}}{\rho_g \mu_g}\right)^{\frac{1}{q}} \frac{(\gamma_{gi}^q + \gamma_{gj}^q)^{\frac{2}{q}}}{\gamma_{gi}^q \gamma_{gj}^q} = \left(\frac{q}{\rho_g \mu_g}\right)^{\frac{1}{q}} \frac{(\gamma_{gi}^q + \gamma_{gj}^q)^{\frac{2}{q}}}{\gamma_{gi}^{1-\frac{1}{q}} \gamma_{gj}^q}.$$

*If $\varepsilon^*_{gi} < 1$ for all g and i then there exists a setting of parameters for which $(\varepsilon^*_{gi}, \varepsilon^*_{gj})$ is the unique equilibrium.*

For brevity, we relegate the full proof and the characterization of when these conditions hold, to Section C. Below, we detail the instructive portion of the proof for the special case in which $q = \rho = 1$.

Lemma 5.7. *Suppose two firms with learning rate* 1 *face proportional demand with competition exponent* 1. *In any interior equilibrium, error levels are given by:*

$$\varepsilon^*_{gi} = \frac{1}{\mu_g} \cdot \frac{(\gamma_{gi} + \gamma_{gj})^2}{\gamma_{gj}}.$$

*Sufficient conditions for existence are that for all g and i: $\phi_{gi} \leq \frac{\mu_g \gamma_{gj}^2}{(\gamma_{gi}+\gamma_{gj})^2}$, $\varepsilon^*_{gi} < 1$, $\frac{(\gamma_{gi}+\gamma_{gj})^2}{\mu_g \gamma_{gj}} < 1$. If, moreover, $\min_k \gamma_{gk} \geq (\max_k \gamma_{gk})^2$, then $(\varepsilon^*_{gi}, \varepsilon^*_{gj})$ is the unique equilibrium.*

The conditions of Lemma 5.7 are stated in terms of $\gamma_{gi}$; recall, though, that $\gamma_{gi}$ is not a primitive of our model, but rather the product of the per-datapoint cost and learning theory constants. These conditions thus imply conditions on these underlying constants. In the symmetric case, this asks that the per-datapoint cost $c_g$ satisfies:

$$c_g \leq \frac{d_H + \log \frac{1}{\delta}}{12^q}.$$

which merely requires that the per-datapoint cost is not too large relative to the desired hypothesis class and success probability. In the asymmetric case, we require that firms do not face ratios of data cost to learning constant that are *too different* from each other. If either of these conditions is violated, then one or both firms may have an incentive to stop investing completely in data acquired for a group. Such non-interior equilibria can obviously lead to severe error inequality, but again, Theorem 5.5 demonstrates the existence of incentives to unfairness *even ruling out these extreme cases.*

Proof of Lemma C.1. We can write Firm $i$'s profit from each group as:

$$\pi_{gi}(\varepsilon_{gi}, \varepsilon_{gj}) = \mu_g \frac{\varepsilon_{gj}}{\varepsilon_{gi} + \varepsilon_{gj}} - \frac{\gamma_{gi}}{\varepsilon_{gi}} - \phi_{gj}.$$

The strategy space of the firm is to select an $\varepsilon_{gi}$ for each group in $(0, 1]$; we search for a pure strategy Nash equilibrium. At a high level, our strategy to do so is as follows: first, we fix the opposing firm's action $\varepsilon_{gj}$. Optimizing Firm i's profit gives a best-response to the fixed action $\varepsilon_{gj}$. An equilibrium pair must simultaneously satisfy *both* firms' first order conditions, given the other, so we obtain two simultaneous equations that yield the equilibrium relationship between the two firms' actions. Solving this yields a candidate solution. Then, we can show that the candidate solution is indeed a maximum via the concavity of the profit function. Finally, we need but check that there are no solutions at the endpoints, and we provide conditions when this is ruled out.

Now, fixing Firm $j$'s choice $\varepsilon_{gj}$, the profit function $\pi_g$ is just a function of $\varepsilon_{gi}$. Differentiating this gives:

$$\frac{\partial \pi_{gi}}{\partial \varepsilon_{gi}} = -\varepsilon_{gj}\mu_g(\varepsilon_{gi}+\varepsilon_{gj})^{-2} + \gamma_{gi}\varepsilon_{gi}^{-2}.$$

We set this equal to zero. Since satisfying this condition is required for $\varepsilon_{gi}$ to be a best-response we can plug in $\varepsilon^*_{gj}$, whatever that may be, requiring:

$$\frac{\varepsilon_{gi}^2\varepsilon^*_{gj}}{\mu_g\gamma_{gi}} = (\varepsilon_{gi}+\varepsilon^*_{gj})^2, \tag{1}$$

and in particular, this must apply to the best-response $\varepsilon^*_{gi}$. We can apply similar logic to Firm $j$. Hence, for $(\varepsilon^*_{gi}, \varepsilon^*_{gj})$ to be best-responses to each other – that is, to be in (interior) equilibrium – we must have that

$$\frac{{\varepsilon^*_{gi}}^2\varepsilon^*_{gj}}{\mu_g\gamma_{gi}} = (\varepsilon^*_{gi}+\varepsilon^*_{gj})^2 = \frac{{\varepsilon^*_{gj}}^2\varepsilon^*_{gi}}{\mu_g\gamma_{gj}}. \tag{2}$$

This implies that

$$\varepsilon^*_{gj} = \varepsilon^*_{gi}\frac{\gamma_{gj}}{\gamma_{gi}}.$$

Substituting this condition back into Equation 2, we obtain that

$$\frac{{\varepsilon^*_{gi}}^3\frac{\gamma_{gj}}{\gamma_{gi}}}{\mu_g\gamma_{gi}} = (\varepsilon^*_{gi}+\varepsilon^*_{gi}\frac{\gamma_{gj}}{\gamma_{gi}})^2 \implies \varepsilon^*_{gi} = \frac{(\gamma_{gi}+\gamma_{gj})^2}{\mu_g\gamma_{gj}}.$$

Symmetric logic yields $\varepsilon^*_{gj} = \frac{(\gamma_{gi}+\gamma_{gj})^2}{\mu_g\gamma_{gi}}$.

Now, to show that this candidate solution is indeed an equilibrium, we must show that these actions are best-responses to each other. Fix $\varepsilon^*_{gj} = \frac{(\gamma_{gi}+\gamma_{gj})^2}{\mu_g\gamma_{gi}}$. Then we can view $\pi_{gi}(\varepsilon_{gi}, \varepsilon^*_{gj})$ as a continuous function on $(0, 1]$. By construction, evaluating $\frac{\partial}{\partial\varepsilon_{gi}}\pi_{gi}(\varepsilon_{gi}, \varepsilon^*_{gj})$ at $\varepsilon^*_{gi}$ must give zero. (It is also easy to verify that this is indeed the case.) If $\pi_{i,\varepsilon^*_{gj}}(\varepsilon_{gi})$ is concave at $\varepsilon^*_{gi}$, then that is a local maximum of the profit function (given $\varepsilon^*_{gj}$).

To see that it is concave, note that

$$\frac{\partial^2}{\partial\varepsilon_{gj}^2}\pi_{gi}(\varepsilon_{gi}, \varepsilon_{gj}) = 2\varepsilon_{gj}\mu_g(\varepsilon_{gj}+\varepsilon_{gi})^{-3} - 2\frac{\gamma_{gi}}{\varepsilon_{gi}^3}.$$

Evaluating this quantity at $\varepsilon^*_{gi}$ gives:

$$\begin{aligned}&\frac{\partial^2}{\partial\varepsilon_{gi}^2}\pi_{gi}(\varepsilon_{gi}, \varepsilon^*_{gj})\Big|_{\varepsilon_{gi}=\varepsilon^*_{gi}} \\ &= 2\mu_g\frac{(\gamma_{gi}+\gamma_{gj})^2}{\mu_g\gamma_{gi}}\left(\frac{(\gamma_{gi}+\gamma_{gj})^2}{\mu_g\gamma_{gj}}+\frac{(\gamma_{gi}+\gamma_{gj})^2}{\mu_g\gamma_{gi}}\right)^{-3} \\ &\quad - 2\frac{\gamma_{gi}}{((\gamma_{gi}+\gamma_{gj})^2/(\mu_g\gamma_{gi}))^3}.\end{aligned}$$

Straightforward, if tedious, algebra lets us rewrite the right hand side and conclude that

$$\frac{\partial^2}{\partial\varepsilon_{gi}^2}\pi_{gi}(\varepsilon_{gi}, \varepsilon^*_{gj})\Big|_{\varepsilon_{gi}=\varepsilon^*_{gi}} = \frac{2\mu_g^3\gamma_{gj}^3\gamma_{gi}}{(\gamma_{gi}+\gamma_{gj})^6}\left[\frac{\gamma_{gi}}{\gamma_{gi}+\gamma_{gj}} - 1\right].$$

But notice that this quantity is always negative if costs are positive; hence, $\varepsilon^*_{gi}$ is indeed a local maximum of $\pi_{i,\varepsilon^*_{gj}}$.

To ensure that this point is a global maximum, we must compare it with the profit at the endpoint. For brevity, we defer this calculation to the Appendix in Section C

Finally, note that equilibrium profits are positive if $\pi_{gi}(\varepsilon^*_{gi}, \varepsilon^*_{gj}) \geq 0$; this is true whenever

$$\frac{\mu_g\gamma_{gj}^2}{(\gamma_{gi}+\gamma_{gj})^2} \geq \phi_{gi}, \tag{3}$$

i.e. fixed costs are not extremely large. Positive profits and the fact that $\varepsilon^*_{gi}$ globally maximizes profit given $\gamma^*_{gj}$ ensures that the putative equilibrium pair forms an equilibrium.

To identify conditions in which this equilibrium is unique, we need to eliminate the only other possible equilibrium (both firms choosing $\varepsilon = 1$). Again, for brevity, we defer this calculation to the appendix.

□

Again, we pause to highlight several intuitive properties of the equilibrium. First, Firm $i$'s choice of error for group $g$ is *decreasing* with the market size of Group $g$ as well as the *ferocity* of competition in Group $g$. These results are similar to those of Lemma 4.3, with a different functional form and the competition exponent of the Tullock game replacing the error elasticity of demand. It is also, intuitively, increasing in $\gamma_{gi}$ and decreasing in $\gamma_{gj}$, though this is harder to see due to the functional form of $f$.

## 5.3 Approximately Rational Demand

Now we consider markets where consumers behave rationally. If we allow consumers to behave *fully* rationally, in the sense of always picking the firm with (even infinitesimally) smaller error, we obtain a model similar to the Bertrand model of competition [19]; accordingly, no equilibrium exists, as we show in Section B.3. Hence, we instead consider a slight relaxation of the fully rational model: Suppose consumers behave rationally, *except* that they do not care about excess error up to $\zeta_g$ over the optimal error. That is, the lower firm will capture the whole market for errors that are not too small, but for $\varepsilon_{gi}, \varepsilon_{gj} \in [0, \zeta_g]$, firms again split the market.

We formally define this demand function below:

Definition 5.8 (Bertrand-like Tolerant Demand). *In a multi-firm market, we say demand is* $\zeta$-tolerant rational *with* $\zeta > 0$ *if*

$$D_{gi}(\varepsilon) = \begin{cases} 1 & \min_k \varepsilon_{gk} > \zeta_g \text{ and } \varepsilon_{gi} < \min_{j\neq i}\varepsilon_{gj} \\ \frac{1}{\sum_j \mathbf{1}[\varepsilon_{gj} = \min_k \varepsilon_{gk}]} & \min_k \varepsilon_{gk} > \zeta_g \text{ and } \varepsilon_{gi} = \min_{j\neq i}\varepsilon_{gj} \\ 0 & \min_k \varepsilon_{gk} > \zeta_g \text{ and } \varepsilon_{gi} > \min_{j\neq i}\varepsilon_{gj} \\ \frac{\mathbf{1}[\varepsilon_{gi}\leq\zeta_g]}{\sum_j \mathbf{1}[\varepsilon_{gj}\leq\zeta_g]} & \min_k \varepsilon_{gk} \leq \zeta_g \end{cases}.$$

We will show that there exists a unique equilibrium here (for appropriate parameters) in which groups' error levels are determined not by their sizes, but by their optimal errors and their tolerances.

Theorem 5.9 (Approximately Rational Inequality). *Suppose that two firms compete under* $\zeta$*-tolerant demand. Then in any interior* *equilibrium, error inequality is given by*

$$\frac{\varepsilon_g}{\varepsilon_{g'}} = \frac{\zeta_g}{\zeta_{g'}}$$

*where $\zeta_g, \zeta_{g'}$ is users' tolerance threshold (assumed to be strictly positive). Moreover, if $\gamma_{gi} < \frac{\zeta_g \mu_g}{2}$ for all $g, i$, the unique equilibrium is interior.*

In particular, Theorem 5.9 shows that under this approximate Bertrand-like model of competition, the dependence on group size in the error inequality is eliminated. Instead, inequality depends merely on the optimal error achievable under the hypothesis class used by firms and groups' tolerances.

Note that the conditions of Theorem 5.9 is just asking that

$$c_{gi} \leq \left(\frac{\mu_g \zeta_g}{2}\right)^q \frac{1}{d_{H_i} + \log \frac{1}{\delta}}.$$

As before, we can interpret this as a condition that the per-datapoint cost is not too large relative to the total market size and the learning theory constants.

Theorem 5.9 follows from the following lemma:

Lemma 5.10 (Approximate Rational Equilibrium). *Suppose that two firms compete under $\zeta$-tolerant demand, and $\gamma_{gi} < \frac{\zeta_g \mu_g}{2}$ for all $g, i$. Then an interior pure strategy equilibrium exists in which*

$$\varepsilon^*_{gi} = \zeta_g,$$

*and this equilibrium is unique.*

Proof. We posit that the profile $(\zeta_g, \zeta_g)$ is an equilibrium. To see this, note that a firm deviating to some $\varepsilon > \zeta_g$ would lose its entire market share, and so would end up with negative profit. Under the conditions of the theorem, though,

$$\pi_i(\zeta_g, \zeta_{g'}) = \frac{\mu_g}{2} - \frac{\gamma_i}{\zeta_g} > 0$$

so deviating to a higher error, with negative profit, cannot be a profitable deviation. On the other hand, deviating to $\varepsilon \in [0, \zeta_g)$ would result in the same market share, but with increased costs. Hence, deviating to decreased error is also not a profitable deviation.

To see that there can be no other equilibria, notice that if both firms were setting error in $\in [0, \zeta_g)$, they would have an incentive to deviate to $\zeta_g$; if one firm's error were in that range and the other's were above, then the firm with higher error would have an incentive to deviate to $\zeta_g$; and finally, if both firms were above $\zeta_g$, either firm could profitably deviate to slightly lower error. □

Unfortunately, even this relaxation of full rationality may not be a realistic model of competition in many cases; it still requires that outside of the range of $[0, \zeta_g]$, all consumers are perfectly discerning. This is unlikely to be true in practice. Without such an assumption, the conclusions of this model do not hold. Models like the proportional split and multilinear demand are more likely to capture salient market features in practice.

# 6 REGULATION

In this section, we consider the perspective of a regulator with the power to require one of two kinds of error equality, and analyze the response of the monopolistic firm to each. These constraints that the regulator may impose are *relative error equality* and *absolute error equality*. We quantify the direct cost associated with imposing these constraints, in terms of increased error to the majority group under the first kind and lost profit to the monopolist in both. This serves to give a sense of the direct tradeoffs involved in regulating machine-learning driven markets. We highlight, though, that there may be non-quantifiable benefits to equity across groups, and only societal deliberation can evaluate these tradeoffs.

Which of these two types of regulation is preferred will depend on the context. Requiring errors across groups to all be similar – relative error equality – may not be sufficiently strong if large error is harmful regardless of another group's error rate, but also may be too strict when small absolute errors are perceived as approximately equivalent. On the other hand, absolute error equality – where we require all errors to be below a threshold – treats all small absolute errors as equivalent, but still allows a large relative gap in error rates across groups. An absolute error bound shifts the 'burden' of fairness entirely to the firm, which may be preferable from a consumer standpoint; at the same time, decreasing profits for monopolies may reduce the incentive to innovate, which may also be undesirable.

We make the following assumption for the rest of the section for ease of exposition:

Assumption 6.1. *There are two groups $\mathcal{G} = \{A, B\}$, there is an interior optimum $\varepsilon^M_A, \varepsilon^M_B < 1$ (i.e. the unconstrained monopoly enters the market), and $B$ has lesser market power and higher data costs, i.e.*

$$\mu_B \beta_B \leq \mu_A \beta_A \qquad \textit{and} \qquad \gamma_B \geq \gamma_A.$$

We refer to group $A$ as the *majority* group and $B$ as the *minority* group. We also define $(\varepsilon^M_A, \varepsilon^M_B)$ and $(\varepsilon^R_A, \varepsilon^R_B)$ to be the monopolist's and regulated monopolist's optimal choices, respectively.

Note that an immediate consequence of Assumption 6.1 and Theorem 4.3 is that $\varepsilon^M_B \geq \varepsilon^M_A$. Finally, we defer omitted proofs from this section to Sections D and E.

## 6.1 Relative Error Equality

In this section, we imagine that a regulator requires the monopolist to achieve error rates within a bounded ratio. We will show that a monopoly responds by investing less in majority data collection and more in minority data collection than it otherwise would, resulting in worse error rates for the majority, better error rates for the minority, approximate equality between groups, and lower profits for the firm. In particular we quantify by how much error rates worsen for the majority and by how much profits are lowered for the monopolist, which we refer to as the 'price' of fairness.

We formalize the regulator's constraint as follows:

Definition 6.2 (Relative error equality). *The regulator forces the firm to achieve error guarantees of bounded ratio:*

$$\frac{\varepsilon_A}{\varepsilon_B} \leq 1 + \chi \qquad \textit{and} \qquad \frac{\varepsilon_B}{\varepsilon_A} \leq 1 + \chi$$

*where $\chi$ is a positive constant.*

As in Section 4, we consider a profit-maximizing monopolist. As before, each group has linear demand with market sizes $\mu_A$ and $\mu_B$.

Now, if the regulation has 'bite' – that is, if it changes the outcome – the regulated monopolist does the minimum it can to satisfy the constraint; that is, it sets $\varepsilon_B^R = \varepsilon_A^R(1+\chi)$. Formally:

Lemma 6.3 (Saturation). *Suppose that the unregulated monopoly sets $\varepsilon_B^M > \varepsilon_A^M(1+\chi)$. Then the profit-maximizing monopoly facing the relative error constraint sets*

$$\varepsilon_B^R = \varepsilon_A^R(1+\chi).$$

The proof follows from concavity and Jensen's inequality; we provide details in D.

Lemma 6.3 allows us to characterize the regulated monopolist's optimal choice of errors under this regulation:

Theorem 6.4. *Suppose that the unregulated monopoly sets error $\varepsilon_B^M > (1+\chi)\varepsilon_A^M$. Then in any interior optimum, the regulated monopoly sets the errors as*

$$\varepsilon_A^R = \left(q \cdot \frac{\gamma_A + \gamma_B/(1+\chi)^q}{\mu_A\beta_A + \mu_B\beta_B(1+\chi)}\right)^{\frac{1}{q+1}}$$

*and $\varepsilon_B^R = (1+\chi)\varepsilon_A^R$.*

Proof. By Lemma 6.3, $\varepsilon_B^R = (1+\chi)\varepsilon_A^R$. Thus, the profit maximization problem can be written solely as a function of $\varepsilon_A$:

$$\pi(\varepsilon_A) = \mu_A(\alpha_A - \beta_A\varepsilon_A) + \mu_B\left(\alpha_B - \beta_B\varepsilon_A(1+\chi)\right) - \left(\phi_A + \frac{\gamma_A}{\varepsilon_A^q}\right) - \left(\phi_B + \frac{\gamma_B}{\varepsilon_A^q(1+\chi)^q}\right).$$

Then, the first order condition is

$$\mu_A\beta_A + \mu_B\beta_B(1+\chi) = \frac{q\left(\gamma_A + \gamma_B/(1+\chi)^q\right)}{\varepsilon_A^{q+1}},$$

and hence we must have that

$$\varepsilon_A^R = \left(\frac{q\left(\gamma_A + \gamma_B/(1+\chi)^q\right)}{\mu_A\beta_A + \mu_B\beta_B(1+\chi)}\right)^{\frac{1}{q+1}}.$$

Concavity guarantees that this is a global optimum. □

These together provide insight into to what the regulation is doing. The monopolist's problem can be written as:

$$\max_\varepsilon \pi(\varepsilon) = \max_\varepsilon \mu_A\alpha_A + \mu_B\alpha_B - (\mu_A\beta_A + \mu_B\beta_B(1+\chi))\varepsilon - (\phi_A + \phi_B) - \frac{1}{\varepsilon^q}(\gamma_A + \gamma_B/(1+\chi)^q).$$

This is equivalent to facing a *single* population of with demand function $\mu_A\alpha_A + \mu_B\alpha_B - (\mu_A\beta_A + \mu_B\beta_B(1+\chi))\varepsilon$, fixed cost $\phi_A + \phi_B$, and marginal cost $(\gamma_A + \gamma_B/(1+\chi)^q)$. We later use this interpretation to quickly calculate the constrained monopolist's profits.

One might worry that imposing fairness requires making both groups worse off in an absolute sense. It turns out that this is not the case; if the regulation has bite, then it necessarily increases the error of the majority group, *and necessarily decreases* the error of the minority group. That is, equality comes at a price for the majority group, but *does not* require a Pareto deterioration.

Our first result is that the monopolist will respond to regulation by increasing majority error rates.

Corollary 6.5.

$$\varepsilon_A^R \geq \varepsilon_A^M \qquad \textit{and} \qquad \varepsilon_B^R \leq \varepsilon_B^M.$$

At this point, members of the majority group may be concerned because their error rate increases. We refer to the gap between their error rates under the constrained and unconstrained monopolies as a *price of fairness* for this reason, even though imposing this constraint may be on the whole desirable from a societal perspective:

$$\mathrm{PoF}_{1+\chi} = \frac{\varepsilon_A^R}{\varepsilon_A^M}.$$

Fortunately, we can show this price is relatively small:

Corollary 6.6 (Price of Fairness Upper Bound).

$$PoF_{1+\chi} \leq \left(1 + \frac{\gamma_B}{\gamma_A}\cdot\frac{1}{(1+\chi)^q}\right)^{\frac{1}{q+1}}.$$

Unsurprisingly, this bound is increasing in the ratio of minority cost to majority cost and decreasing in the leniency of the regulator. Also unsurprisingly, decreasing the ratio $\frac{\mu_B}{\mu_A}$ or $\frac{\beta_B}{\beta_A}$ and increasing the ratio $\frac{\gamma_B}{\gamma_A}$ all increase the price of fairness for the majority.

If regulation changes the monopolist's behavior, it must weakly decrease profits. This loss is quantifiable as another price of fairness:

Definition 6.7 (Monopolist Price of Fairness, Relative Error). *We define the price of fairness as the ratio between the unconstrained monopoly profit and constrained monopoly profit under the relative error constraint, i.e.*

$$MonPoF_{1+\chi} = \frac{\pi\left(\varepsilon_A^M, \varepsilon_B^M\right)}{\pi\left(\varepsilon_A^R, \varepsilon_B^R\right)} = \frac{\pi(\varepsilon_A^M, \varepsilon_B^M)}{\max_{\varepsilon_A,\varepsilon_B:\frac{\varepsilon_A}{\varepsilon_B}\leq 1+\chi,\ \frac{\varepsilon_B}{\varepsilon_A}\leq 1+\chi}\pi(\varepsilon_A,\varepsilon_B)}.$$

We can write down this price of fairness as a function of the parameters of the model:

Theorem 6.8. *The Monopolist's price of fairness is given by*

$$MonPoF_{1+\chi} = \frac{\mu_A\alpha_A + \mu_B\alpha_B - Q(\mu_A\beta_A)^{\frac{q}{q+1}}\gamma_A^{\frac{1}{q+1}} - Q(\mu_B\beta_B)^{\frac{q}{q+1}}\gamma_B^{\frac{1}{q+1}}}{\mu_A\alpha_A + \mu_B\alpha_B - Q(\mu_A\beta_A + \mu_B\beta_B(1+\chi))^{\frac{q}{q+1}}(\gamma_A + \gamma_B\frac{1}{(1+\chi)^q})^{\frac{1}{q+1}}},$$

*where $Q = q^{\frac{1}{q+1}} + \frac{1}{q^{q/(q+1)}}$.*

Proof. The optimal solution to the monopolist's problem with parameters $\mu_g, \alpha_g, \beta_g, \gamma_g$ for $g$ in $\mathcal{G} = \{A, B\}$ is the following:

$$\pi^*(\varepsilon^*) = \sum_{g\in\{A,B\}} \mu_g\alpha_g - (\mu_g\beta_g)^{\frac{q}{q+1}}\gamma_g^{\frac{1}{q+1}}Q.$$

(See Appendix D.) Using this form and plugging in the market parameters, we obtain the optimal profit of the unconstrained monopolist for the numerator. The denominator is derived using the interpretation of the constrained monopolist's problem as optimizing its profits against a single market with parameters modified by regulation, and plugging these parameters into the same form. □

Theorem 6.8 provides a quantitative *price of fairness* in terms of monopoly profits. However, it is somewhat unwieldy; Proposition 6.9 provides some clarity on the limiting behavior of this price of fairness as a function of the minority group's size in absolute terms.

Proposition 6.9 (MonPoF Limit - Relative Error Inequality). *Let $\mu_B/\mu_A = r$ for constant ratio $r$. Then*

$$\lim_{\mu_B \to \infty} MonPoF_{1+\chi} = 1.$$

*On the other hand, for constant $\mu_A$,*

$$\lim_{\mu_B \to 0} MonPoF_{1+\chi} = \frac{1-(Q/q)\varepsilon_A^M}{1-(Q/q)\varepsilon_A^M\left[1+\frac{\gamma_B}{\gamma_A}\frac{1}{(1+\chi)^{\frac{1}{q+1}}}\right]^{\frac{1}{q+1}}}$$

*where $Q$ is as above.*

## 6.2 Absolute Error Equality

In this section, we suppose instead that the regulator imposes an absolute upper bound on error rates for each group. We show that the monopolist responds by purchasing just enough data to meet the constraint using the profits from the majority to subsidize the minority. In this case, minority error rates can be improved without increasing error for the majority; the regulator can even improve error rates for the majority as well, up to a point. We characterize the price of fairness for the monopolist and the minimum error the regulator can guarantee. We formalize this constraint as follows:

Definition 6.10 (Absolute error equality). *For $\chi < 1$, the regulator forces the firm to achieve error of:*

$$\varepsilon_A \leq \chi \qquad \text{and} \qquad \varepsilon_B \leq \chi.$$

We have another saturation lemma for this kind of constraint too: either the unconstrained error was already less than $\chi$, or the profit maximizing error subject to regulation is exactly $\chi$. Formally:

Lemma 6.11 (Saturation). *$\forall g \in \{A, B\}$, if $\varepsilon_g^R \neq \varepsilon_g^M$ then $\varepsilon_g^R = \chi$.*

Lemma 6.11 lets us reason very simply about the behavior of the regulated monopolist: for any group in which imposing regulation requires the firm to improve error rates, the firm will use up the entirety of this 'error budget.' Profit will decrease, of course, because imposing constraints can only decrease its objective. In this scenario, if the firm enters the market at all, it must enter the market for both groups so as to achieve the constrained error rates. A regulator then has to choose $\chi$ so as to still induce the firm to enter the market at all if they want to ensure the constrained error rate for the minority group. Of course, a regulator may also wish to choose the smallest such error rate, which we refer to as the *minimum achievable error.* Lemma 6.11 let us characterize the minimum achievable error:

Proposition 6.12. *Let $\chi_0$ be the smallest $\chi \in [0, 1]$ which solves*

$$K_1\chi^{q+1} + K_2\chi^q - K_3 = 0, \tag{4}$$

*where $K_1 = -(\mu_A\beta_A + \mu_B\beta_B)$, $K_2 = \mu_A\alpha_A + \mu_B\alpha_B - \phi_A - \phi_B$, and $K_3 = \gamma_A + \gamma_B$. $\chi_0$ exists and is the minimum achievable error, i.e. the minimum $\chi \in [0, 1]$ for which the monopolist still enters the market.*

Equation 4 can be solved via the quadratic or cubic formulae in the realizable and agnostic cases, respectively, and learning rates in between can be accommodated numerically. This leads us to the monopoly's optimal error rates as a function of $\chi$:

Theorem 6.13 (Absolute Outcomes). *Outcomes fall into one of the following possibilities:*

1. *If $\chi \geq \varepsilon_B^M$ then $(\varepsilon_B^R, \varepsilon_B^R) = (\varepsilon_A^M, \varepsilon_B^M)$.*
2. *If $\varepsilon_A^M \leq \chi < \varepsilon_B^M$ then $(\varepsilon_B^R, \varepsilon_B^R) = (\varepsilon_A^M, \chi)$.*
3. *If $\chi_0 < \chi < \varepsilon_A^M$ then $(\varepsilon_B^R, \varepsilon_B^R) = (\chi, \chi)$.*
4. *If $\chi < \chi_0$ then the firm exits the market.*

Proof. Case 1 is trivial. Case 2 and 3 follow from Lemma 6.11. Case 4 follows by the definition of $\chi_0$. □

Theorem 6.13 contrasts starkly with Theorem 6.5: as long as the constraint is not so strict the monopolist exits the market, outcomes either improve for the minority and remain just as good for the majority, or improve for *both* groups. In other words, this style of regulation *does not impose a price of fairness on the majority.* Note that unless $\varepsilon_0 < \chi < \varepsilon_A^M$, the regulator is *not* guaranteeing relative equality. Which type of equality is preferable will depend on the context. Of course, this regulation *does* still impact profit:

Definition 6.14 (Monopolist Price of Fairness). *We define the monopolist's* price of fairness *under absolute error constraints as:*

$$MonPoF_\chi = \frac{\pi\left(\varepsilon_A^M, \varepsilon_B^M\right)}{\pi\left(\varepsilon_A^R, \varepsilon_B^R\right)} = \frac{\pi(\varepsilon_A^M, \varepsilon_B^M)}{\max_{\varepsilon_A, \varepsilon_B : \varepsilon_A \leq \chi, \varepsilon_B \leq \chi} \pi(\varepsilon_A, \varepsilon_B)}.$$

Notice that given the market parameters, Theorem 6.13 allows the regulator to evaluate the monopolist's price of fairness for each potential choice of error threshold via straightforward calculation. Proposition 6.15 characterizes the limiting behavior of the monopolist's price of fairness as a function of absolute size of the minority group under absolute error guarantees, and these are qualitatively similar to limiting behavior under relative error guarantees.

Proposition 6.15 (MonPoF Limit - Absolute Error Guarantees). *For* fixed *$\chi$, and for $\mu_B \to \infty$ at a constant ratio $\mu_A/\mu_B = r$:*

$$\lim_{\mu_B \to \infty} MonPoF_\chi = 1.$$

*On the other hand, let $\chi_0$ be the minimal achievable error when $\mu_B = 0$ (i.e. when the firm faces group A alone). Then if $\chi > \chi_0$, then $MonPoF_\chi$ converges to a parameter-specific constant as $\mu_B \to 0$.*

# 7 DISCUSSION

In this work, we identify economic incentives leading to unfairness in data-driven markets. At a high level, we show that monopolists are incentivized to invest less in minority groups (as measured by market size, elasticity, and data costs) because they are less profitable; that competition does not mitigate this incentive towards inequality, under reasonable models; and that judicious regulation *can* improve outcomes, potentially at a cost in terms of profits or, depending on the regulation, error rates for the majority group.

We view this paper as highlighting an important and understudied point of view, but certainly not as the last word. We made many choices that situate our models in particular contexts; for example, the assumption that firms and users benefit from improved accuracy does not capture many settings that currently are or will soon be urgent domains of adjudicating fairness concerns - machine learning in loans, insurance, and facial recognition systems are obvious cases, but the potential, and consequent scope for unfairness, is vast. We hope that future work will further clarify the possibility - and perhaps necessity- of leveraging policy tools in addition to algorithmic solutions to combat unfairness in machine learning.

# REFERENCES


[1] Guy Aridor, Kevin Liu, Aleksandrs Slivkins, and Zhiwei Steven Wu. 2019. Competing Bandits: The Perils of Exploration under Competition. *arXiv preprint arXiv:1902.05590* (2019).

[2] Rajiv D. Banker, Inder Khosla, and Kingshuk K. Sinha. 1998. Quality and Competition. *Manage. Sci.* 44, 9 (Sept. 1998), 1179–1192. https://doi.org/10.1287/mnsc.44.9.1179

[3] Omer Ben-Porat and Moshe Tennenholtz. 2018. Competing prediction algorithms. *arXiv preprint arXiv:1806.01703* (2018).

[4] Omer Ben-Porat and Moshe Tennenholtz. 2019. Regression equilibrium. *arXiv preprint arXiv:1905.02576* (2019).

[5] Richard Berk, Hoda Heidari, Shahin Jabbari, Michael Kearns, and Aaron Roth. 2018. Fairness in criminal justice risk assessments: The state of the art. *Sociological Methods & Research* (2018), 0049124118782533.

[6] Steven T. Berry. 1994. Estimating Discrete-Choice Models of Product Differentiation. *The RAND Journal of Economics* 25, 2 (1994), 242–262. http://www.jstor.org/stable/2555829

[7] Reuben Binns. 2017. Fairness in machine learning: Lessons from political philosophy. *arXiv preprint arXiv:1712.03586* (2017).

[8] Su Lin Blodgett and Brendan O'Connor. 2017. Racial disparity in natural language processing: A case study of social media african-american english. *arXiv preprint arXiv:1707.00061* (2017).

[9] Joy Buolamwini and Timnit Gebru. 2018. Gender shades: Intersectional accuracy disparities in commercial gender classification. In *Conference on Fairness, Accountability and Transparency*. 77–91.

[10] Irene Chen, Fredrik D Johansson, and David Sontag. 2018. Why is my classifier discriminatory?. In *Advances in Neural Information Processing Systems*. 3539–3550.

[11] Bo Cowgill and Catherine E Tucker. 2019. Economics, fairness and algorithmic bias. *preparation for: Journal of Economic Perspectives* (2019).

[12] Jinshuo Dong, Hadi Elzayn, Shahin Jabbari, Michael Kearns, and Zachary Schutzman. 2019. Equilibrium Characterization for Data Acquisition Games. *Proceedings of the International Joint Conference on Artificial Intelligence*.

[13] Michael D Ekstrand, Mucun Tian, Ion Madrazo Azpiazu, Jennifer D Ekstrand, Oghenemaro Anuyah, David McNeill, and Maria Soledad Pera. 2018. All the cool kids, how do they fit in?: Popularity and demographic biases in recommender evaluation and effectiveness. In *Conference on Fairness, Accountability and Transparency*. 172–186.

[14] Hadi Elzayn, Shahin Jabbari, Christopher Jung, Michael Kearns, Seth Neel, Aaron Roth, and Zachary Schutzman. 2019. Fair algorithms for learning in allocation problems. In *Proceedings of the Conference on Fairness, Accountability, and Transparency*. ACM, 170–179.

[15] Danielle Ensign, Sorelle A Friedler, Scott Neville, Carlos Scheidegger, and Suresh Venkatasubramanian. 2017. Runaway feedback loops in predictive policing. *arXiv preprint arXiv:1706.09847* (2017).

[16] Benjamin Fish, Ashkan Bashardoust, Danah Boyd, Sorelle A. Friedler, Carlos Scheidegger, and Suresh Venkatasubramanian. 2019. Gaps in Information Access in Social Networks. In *The World Wide Web Conference, WWW 2019, San Francisco, CA, USA, May 13-17, 2019*. 480–490.

[17] D. Fudenberg, J. Tirole, J.A. TIROLE, and MIT Press. 1991. *Game Theory*. MIT Press. https://books.google.ca/books?id=pFPHKwXro3QC

[18] F. Hayashi. 2011. *Econometrics*. Princeton University Press. https://books.google.com/books?id=QyIW8WUIyzcC

[19] Kenneth L Judd. 1996. Cournot versus Bertrand: A dynamic resolution. (1996).

[20] Nathan Kallus and Angela Zhou. 2018. Residual unfairness in fair machine learning from prejudiced data. *arXiv preprint arXiv:1806.02887* (2018).

[21] Michael Kearns. 2017. Fair Algorithms for Machine Learning. In *Proceedings of the 2017 ACM Conference on Economics and Computation (EC '17)*. ACM, New York, NY, USA, 1–1. https://doi.org/10.1145/3033274.3084096

[22] M. Kearns and A. Roth. 2019. *The Ethical Algorithm: The Science of Socially Aware Algorithm Design*. Oxford University Press, Incorporated. https://books.google.ca/books?id=sE2jwgEACAAJ

[23] Michael J. Kearns and Umesh V. Vazirani. 1994. *An Introduction to Computational Learning Theory*. MIT Press. https://books.google.com/books?id=vCA01wY6iywC

[24] Kristian Lum and William Isaac. 2016. To predict and serve? *Significance* 13, 5 (2016), 14–19.

[25] Yishay Mansour, Aleksandrs Slivkins, and Zhiwei Steven Wu. 2018. Competing Bandits: Learning Under Competition. In *9th Innovations in Theoretical Computer Science Conference (ITCS 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

[26] Rishabh Mehrotra, Ashton Anderson, Fernando Diaz, Amit Sharma, Hanna Wallach, and Emine Yilmaz. 2017. Auditing Search Engines for Differential Satisfaction Across Demographics. In *Proceedings of the 26th International Conference on World Wide Web Companion (WWW '17 Companion)*. International World Wide Web Conferences Steering Committee, Republic and Canton of Geneva, Switzerland, 626–633. https://doi.org/10.1145/3041021.3054197

[27] Ziad Obermeyer and Sendhil Mullainathan. 2019. Dissecting Racial Bias in an Algorithm that Guides Health Decisions for 70 Million People. In *Proceedings of the Conference on Fairness, Accountability, and Transparency*. ACM, 89–89.

[28] Shai Shalev-Shwartz and Shai Ben-David. 2014. *Understanding machine learning: From theory to algorithms*. Cambridge university press.

[29] Carl Shapiro. 1989. Theories of oligopoly behavior. *Handbook of industrial organization* 1 (1989), 329–414.

[30] Latanya Sweeney. 2013. Discrimination in Online Ad Delivery. *Queue* 11, 3, Article 10 (March 2013), 20 pages. https://doi.org/10.1145/2460276.2460278

[31] Jean Tirole. 1988. *The theory of industrial organization*. MIT press.

[32] Gordon Tullock. 2001. Efficient rent seeking. In *Efficient Rent-Seeking*. Springer, 3–16.

[33] Sahil Verma and Julia Rubin. 2018. Fairness definitions explained. In *Proceedings of the International Workshop on Software Fairness, FairWare@ICSE 2018, Gothenburg, Sweden, May 29, 2018*. 1–7.

[34] Benjamin Wilson, Judy Hoffman, and Jamie Morgenstern. 2019. Predictive Inequity in Object Detection. *CoRR* abs/1902.11097 (2019). arXiv:1902.11097 http://arxiv.org/abs/1902.11097

# A PIECE-WISE LINEAR DEMAND

In this section, we consider the possibility of a *piece-wise* linear demand function. Such a demand has the same spirit of the linear demand function, in that market share declines linearly with worst-case error rate, but allows for a more general parameter range. In particular, a firm with piece-wise linear demand may capture less than the full market (but, logically, not more) with perfect accuracy, and may lose the entire market even at relatively high accuracy. Imposing a cap and floor on a linear demand function whose parameters fall outside the restricted range described in Section 4 allows us to accomplish this.

We formally write piece-wise linear demand as follows:

$$D_g(\varepsilon_g) = \begin{cases} 0 & \text{if } \varepsilon_g \geq \alpha_g/\beta_g \\ \alpha_g - \beta_g \varepsilon_g & \text{if } \frac{\alpha_g - 1}{\beta_g} \leq \varepsilon_g \leq \alpha_g/\beta_g \\ 1 & \text{if } \varepsilon_g \leq \frac{\alpha_g - 1}{\beta_g}, \end{cases}$$

where $\alpha_g, \beta_g > 0$, and $\frac{\alpha_g - 1}{\beta_g} < 1$.

Finding the optimal choice of the monopolist under this demand requires slightly more care than linear demand but is substantively similar. We provide an outline below.

Lemma A.1. *For convenience, let*

$$\tilde{\varepsilon}_g = \max\left\{\min\left\{\left(\frac{q\gamma_g}{\mu_g \beta_g}\right)^{\frac{1}{q+1}}, \frac{\alpha_g}{\beta_g}, 1\right\}, \frac{\alpha_g - 1}{\beta_g}\right\}.$$

*A monopolist, under linear demand enters the market for group g if and only if*

$$\pi_g\left(\tilde{\varepsilon}_g\right) > 0,$$

*and if they do, the equilibrium error rate $\varepsilon_g^*$ is:*

$$\varepsilon_g^* = \tilde{\varepsilon}_g \qquad .$$

Proof outline. Since the profit is additively separable over $g$, we consider each $\pi_g$ separately. For $\varepsilon_g \geq \alpha/\beta$, note profit is always negative. And for $\varepsilon_g \leq \frac{\alpha_g - 1}{\beta_g}$, demand is increasing as $\varepsilon_g$ increases, which can be seen by checking the derivative. Then if profits are positive, $\frac{\alpha_g - 1}{\beta_g} \leq \varepsilon_g^* \leq \alpha_g/\beta_g$. Thus either $\varepsilon^*$ is one of those end points, or $\varepsilon^*$ satisfies the first order condition $\varepsilon_g^* = \varepsilon_g : \frac{\partial \pi}{\partial \varepsilon_g} |_{\varepsilon_g} = 0$, as in Lemma 4.3, and thus $\varepsilon_g^* = \tilde{\varepsilon}_g$.

Moreover, if the maximum profit is positive, it must be attained with $\varepsilon_g^* \leq \alpha_g/\beta_g$, so it must be the case that the profit obtained at $\tilde{\varepsilon}_g$ is positive, and vice versa. □

# B CONSUMER MODELS

In this section, we show how natural models of consumer behavior give rise to the demand functions we assumed for our analysis.

## B.1 Linear Demand

First, consider the following interaction between one firm and a representative user: The firm sets its error levels; the user uses the service if they will receive an accurate answer with probability higher than some threshold corresponding to their outside option (i.e. the payoff they would get if they decide not to use the service). While the user knows her outside option, the firm does not; a standard approach is to assume the firm makes decisions as if the user's outside option were drawn from a *distribution*. If this distribution is *uniform* over some interval, then there is a linear relationship between choice of error and probability (from the firm's perspective) of the user choosing to use the service (and thus the firm's expected revenue). If the firm interacts with many users, and these threshold are uniform throughout the population, then this representative interaction captures the aggregate interaction the firm faces.

We formalize the interaction as follows: A firm provides a service to a *user* wishing to answer some query. If the response is accurate, the user receives a payoff of 1; otherwise, 0. The firm's worst-case error rate $\varepsilon$ is known to the user, and the user chooses whether or not to use the firm's service based on their expected utility under the worst-case error. The user has some parameter, $\tau$, describing their payoff from choosing not to use the service. This parameter is drawn from the uniform distribution over $[\underline{\tau}, \bar{\tau}]$, that describes their outside option distribution.

To see the correspondence between this model and linear demand, we claim that any linear demand function $D(\varepsilon) := \alpha - \beta\varepsilon$ can be mapped to the probability that a user uses the service under some particular choice $[\underline{\tau}, \bar{\tau}]$. Formally:

Proposition B.1. *For any linear demand function $D(\varepsilon) = \alpha - \beta\varepsilon$, there exists a uniform outside option model with choice $\underline{\tau} = 1 - \frac{\alpha}{\beta}$, $\bar{\tau} = 1 + \frac{1-\alpha}{\beta}$ that justifies it.*

Proof. To see this, first note that the user will use the service if and only if the expected payoff is less than his outside option. Since the user receives a payoff of 1 if the service answers correctly and 0 if it answers incorrectly, the expected payoff is merely $1 * \Pr[\text{correct}] + 0 * \Pr[\text{incorrect}] = 1 - \varepsilon$. Hence, the user will use the service if and only if $1 - \varepsilon \geq \tau$. Now, since the user's outside option is, from the Firm's perspective, a uniform random variable, the probability that the user will use the service, as a function of $\varepsilon$, can be written as:

$$\begin{aligned} \Pr[\text{user uses}](\varepsilon) &= \Pr_{\tau \sim U[\alpha, \beta]}[\varepsilon < 1 - \tau] \\ &= \Pr_{\tau \sim U[\alpha, \beta]}[\tau < 1 - \varepsilon] \\ &= \frac{1 - \varepsilon - \underline{\tau}}{\bar{\tau} - \underline{\tau}} = \frac{1 - \underline{\tau}}{\bar{\tau} - \underline{\tau}} - \frac{\varepsilon}{\bar{\tau} - \underline{\tau}}. \end{aligned}$$

Letting $\alpha = \frac{1-\underline{\tau}}{\bar{\tau}-\underline{\tau}}$, $\beta = \frac{1}{\bar{\tau}-\underline{\tau}}$ and solving for $\bar{\tau}$ and $\underline{\tau}$ yields the claim. □

Notice that the truth of the claim is a matter of algebra and holds even beyond sensible choices for $\alpha$ and $\beta$. That is, choosing $\alpha > 1$ would still map to a plausible instance of linear demand, but $\alpha > 1$ would not be sensible as the intercept for a linear probability model. Finally, notice that the simple case of $\alpha = 1, \beta = 1$ corresponds to the uniform random variable over $[0, 1]$.

## B.2 Proportional Split

Consider the following Markov chain representing plausible user behavior in the presence of competition: at any time $t$, a user who is currently using Firm $i$ stays with Firm $i$ into time $t + 1$ if the firm does not make a mistake; otherwise, the user switches to Firm $j$ with probability $\alpha$ and leaves the market with probability $1 - \alpha$. A user outside the market re-enters it with probability $\beta$, and then chooses uniformly from the firms.

The steady state distribution of this Markov chain solves the following equations:

$$\mu_1 = (1-\varepsilon_1)\mu_1 + \alpha\varepsilon_2\mu_2 + \frac{\beta}{2}\mu_3$$
$$\mu_2 = \alpha\varepsilon_1\mu_1 + (1-\varepsilon_2)\mu_2 + \frac{\beta}{2}\mu_3$$
$$\mu_3 = (1-\alpha)\varepsilon_1\mu_1 + (1-\alpha)\varepsilon_2\mu_2 + (1-\beta)\mu_3.$$

Viewing the firm's market share as the proportion of times the user chooses the firm over a long enough horizon (or over many enough consumers) yields a correspondence between the market share and the stationary distribution. The form of this correspondence follows from the following lemma:

Lemma B.2. *Firm i's market share under this Markov process is given by*

$$\mu_i = \frac{\varepsilon_j}{\varepsilon_i + \varepsilon_j + \tau\varepsilon_i\varepsilon_j}.$$

*where* $\tau = 2\frac{1-\alpha}{\beta}$.

Proof. We first the original three equations characterizing the steady state distribution as:

$$\mu_1 = \frac{1}{\varepsilon_1}\left[\alpha\varepsilon_2\mu_2 + \frac{\beta}{2}\mu_3\right]$$
$$\mu_2 = \frac{1}{\varepsilon_2}\left[\alpha\varepsilon_1\mu_1 + \frac{\beta}{2}\mu_3\right]$$
$$\mu_3 = \frac{1-\alpha}{\beta}\left[\varepsilon_1\mu_1 + \varepsilon_2\mu_2\right].$$

Solving the first two equations for $\mu_3$ and setting equal to each other requires that

$$\frac{2\varepsilon_1}{\beta}[\mu_1 - \alpha\frac{\varepsilon_2}{\varepsilon_2}\mu_2] = \frac{2\varepsilon_2}{\beta}[\mu_2 - \alpha\frac{\varepsilon_2}{\varepsilon_2}\mu_1] \implies \varepsilon_1\mu_1 = \varepsilon_2\mu_2.$$

Substituting this into the first rewritten equation for $\mu_3$ gives that

$$\mu_3 = \frac{1-\alpha}{\beta}2\varepsilon_1\mu_1.$$

Finally applying the constraint that $\mu_1 + \mu_2 + \mu_3 = 1$ implies that

$$\mu_1 + \frac{\varepsilon_1}{\varepsilon_2}\mu_1 + 2\varepsilon_1\frac{1-\alpha}{\beta}\mu_1 = 1 \implies \mu_1 = \frac{1}{1 + \frac{\varepsilon_1}{\varepsilon_2} + 2\frac{1-\alpha}{\beta}\varepsilon_1}.$$

Now, we can reparameterize $2\frac{1-\alpha}{\beta}$ as $\tau$, and apply the symmetric logic to the other firm to obtain the general result:

$$\mu_i = \frac{\varepsilon_j}{\varepsilon_i + \varepsilon_j + \tau\varepsilon_i\varepsilon_j}.$$

Thus, viewing the market share of Firm $i$ as its share of the stationary distribution gives the result claimed. □

Notice that the case of $\alpha = 1$ recovers the case in which firms split the complete market, and we can again consider integral competition exponents as requiring $\rho$ mistakes in a row before switching. In this paper, we only consider the case in which $\alpha = 1$.

## B.3 Fully Rational Demand

The Bertrand model of competition considers firms competing on price with *fully rational* consumers. These consumers will always pick the firm with (even infinitesimally) lower price. It is known that a Nash equilibrium exists when firms have identical constant marginal costs in quantity and can produce an unlimited quantity. In that case, firms set equilibrium price equal to marginal cost (that is, the lowest price that firms could charge without losing money). We modify the Bertrand model to apply to our setting. Firms do not set prices in our model; instead, they change error rates. This is not a perfect analogy – changing error rates is itself costly – but captures the spirit of the Bertrand model. However, as we show in this section, equilibrium need not exist in the fully rational model (just as a pure-strategy equilibrium need not exist in canonical Bertrand competition when firms face non-constant marginal costs).

Informally, we say that demand is *fully rational*, or Bertrand-like, if firms with the minimum error capture the entire market (with ties broken by splitting the market equally).

Definition B.3 (Fully Rational Demand). *In a multi-firm market, we say that demand is* fully rational *if*

$$D_{gi}(\varepsilon) = \begin{cases} 1 & \varepsilon_{gi} < \min_{j\neq i} \varepsilon_{gj} \\ \frac{1}{\sum_j \mathbf{1}[\varepsilon_{gj} = \min_k \varepsilon_{gk}]} & \varepsilon_{gi} = \min_{j\neq i} \varepsilon_{gj} \\ 0 & \varepsilon_{gi} > \min_{j\neq i} \varepsilon_{gj} \end{cases}.$$

A proposition we will show is that there is no equilibrium in pure strategies when considering this fully-rational demand.

Proposition B.4. *The game induced by* fully rational demands *as described in Definition B.3 has no equilibrium in pure strategies whenever* $c_{gi} < \frac{\mu_g}{2}$ $\forall i$ *for some group g.*

Proof. Suppose there existed such an equilibrium. Consider a single group and let $(\varepsilon^*_{gi}, \varepsilon^*_{gj})$ be the putative equilibrium error choices. Note that these correspond to equilibrium choices of data $(M^*_{gi}, M^*_{gj})$. We claim that a profitable deviation will exist regardless of what these choices are. There are two cases: in the first, firms have different errors, while in the second, firms have the same error. If firms have different errors, without loss of generality suppose that $\varepsilon^*_{gi} < \varepsilon^*_{gj}$. Then Firm $i$ receives $\mu_g - \gamma_{gi}/\varepsilon^q_{gi} - \phi_{gi}$, while Firm $j$ attains zero revenue. But notice that Firm $i$ can unilaterally deviate to $\varepsilon' \in (\varepsilon^*_{gi}, \varepsilon^*_{gj})$ and capture the full market while paying less, thus improving profits. Hence, we cannot have an equilibrium when firms are choosing different error rates. On the other hand, suppose firms are choosing the same error rates, that is, $\varepsilon^*_{gi} = \varepsilon^*_{gj}$. Now, we can link $\varepsilon^*_{gi}$ to $M^*_{gi}$ via $\varepsilon^*_{gi} = \frac{(d_H + \log\frac{1}{\delta})}{(M^*_{gi})^{\frac{1}{q}}}$. In this case, each firm is earning $\frac{\mu_g}{2} - c_g M^*_{gk} - \phi_{gk}$. Consider Firm $i$ buying an additional data point, i.e. $M'_{gi} = M^*_{gi} + 1$.Then because worst-case error guarantees are strictly decreasing in the number of datapoints purchased, we must have that $\varepsilon'_{gi} < \varepsilon^*_{gi}$, and thus the firm deviating to $M'_{gi}$ would capture the whole market at a cost of $c_{gi}(M^*_{gi} + 1)$. This deviation will be profitable if

$$\mu_g - c_{gi}(M^*_{gi}+1) > \frac{\mu_g}{2} - c_{gi}M^*_{gi} \iff \frac{\mu_g}{2} > c_{gi}.$$

Thus if $c_{gi} < \frac{\mu_g}{2}\ \forall i$, $(\varepsilon^*_{gi}, \varepsilon^*_{gj})$ cannot be an equilibrium. □

A natural way to relax full rationality is to allow consumers to be rational *up to a point*. That is, above some threshold $\xi$, they can perfectly discriminate between error rates, and always will choose the firm with (even infinitesimally) smaller error. But below $\xi$, increasing accuracy does not materially improve their utility of the project, and rather than attempt to ferret out small differences, they pick randomly among firms with error below $\xi$. This leads to our *$\xi$-tolerant rational demand* as discussed in Section 5.3.

# C OMITTED PROOFS FROM SECTION 5

Remainder of Proof of Lemma 5.7. The profit of playing $\varepsilon^*_{gi}$ given that $j$ chooses $\varepsilon^*_{gj}$ is

$$\pi_{gi}(\varepsilon^*_{gi}, \varepsilon^*_{gj}) = \frac{\mu_g \gamma^2_{gj}}{(\gamma_{gi}+\gamma_{gj})^2} - \phi_{gi}.$$

On the other hand, if the firm chooses $\varepsilon_{gi} = 1$, its profit can be upper bounded as:

$$\pi_{gi}(1, \varepsilon^*_{gj}) \le \mu_g \frac{(\gamma_{gi}+\gamma_{gj})^2}{\gamma_{gi} + (\gamma_{gi}+\gamma_{gj})^2} - \phi_{gi}.$$

Hence, their difference is at least:

$$\pi_{gi}(\varepsilon^*_{gi}, \varepsilon^*_{gj}) - \pi_{gi}(1, \varepsilon^*_{gj}) = \ge \mu_g \left[ \frac{\gamma^2_{gj}}{(\gamma_{gi}+\gamma_{gj})^2} - \frac{(\gamma_{gi}+\gamma_{gj})^2}{\gamma_{gi} + (\gamma_{gj}+\gamma_{gi})^2} \right].$$

Thus, a sufficient condition that $\pi_{gi}(\varepsilon^*_{gi}, \varepsilon^*_{gj}) \ge \pi_{gi}(1, \varepsilon^*_{gj})$ is:

$$\frac{(\gamma_{gi}+\gamma_{gj})^2}{\gamma_{gi} + (\gamma_{gi}+\gamma_{gj})^2} < \frac{\gamma^2_{gj}}{(\gamma_{gi}+\gamma_{gj})^2}.$$

This is true if and only if:

$$(\gamma_{gi}+\gamma_{gj})^4 < \gamma^2_{gj}\left[\gamma_{gi} + (\gamma_{gi}+\gamma_{gj})^2\right]. \tag{5}$$

On the other hand, to ensure that $\varepsilon^*_{gj}$ is a best-response to $\varepsilon^*_{gi}$, we carry out the symmetric logic for Firm $j$. This will require that

$$(\gamma_{gi}+\gamma_{gj})^4 < \gamma^2_{gi}[\gamma_{gj} + (\gamma_{gi}+\gamma_{gj})^2]. \tag{6}$$

Both inequalities must be satisfied if our purported equilibrium is to be truly an equilibrium. Characterizing possible simultaneous solutions to Inequalities 5 and 6 is tedious, so instead we note that it suffices to ensure $\min_k \gamma_{gk} \ge 12(\max_k \gamma_{gk})^2$; to avoid encumbering the current argument, we defer the proof of this fact to Appendix C.0.1. These are not the *only* solutions conditions that satisfy Inequality 5, but they are sufficient conditions convenient to write down.

Finally, note that equilibrium profits are positive if $\pi_{gi}(\varepsilon^*_{gi}, \varepsilon^*_{gj}) \ge 0$; this is true whenever

$$\frac{\mu_g \gamma^2_{gj}}{(\gamma_{gi}+\gamma_{gj})^2} \ge \phi_{gi}, \tag{7}$$

i.e. fixed costs are not extremely large.

Thus, $(\varepsilon^*_{gi}, \varepsilon^*_{gj})$ satisfy these three conditions- Inequality, 5, Inequality 6, and Inequality 7 - and $\varepsilon^*_{gi} < 1$; hence $(\varepsilon^*_{gi}, \varepsilon^*_{gj})$ is truly an interior equilibrium.

To identify conditions in which this equilibrium is unique, we need to eliminate the possible equilibrium in which both firms choose $\varepsilon = 1$. Thus, we must show that there exists a choice $\varepsilon_{gi}$ such that:

$$\pi_{gi}(\varepsilon_{gi}, 1) > \pi_{gi}(1,1);$$

that is, if one firm chooses $\varepsilon_g = 1$, and so abandons hope of serving a good model to group $g$, we want to show that there is always a more profitable choice for the other firm than also giving up on them. One possibility, though not necessarily the optimal one, is that the other firm could choose $\varepsilon_{gi} = \varepsilon^*_{gi}$, i.e. the interior equilibrium choice we found above. This being more profitable is asking that $\pi_{gi}(\varepsilon^*_{gi}, 1) - \pi_{gi}(1,1) > 0$. But this requires:

$$\mu_g \frac{1}{1 + \frac{(\gamma_{gi}+\gamma_{gj})^2}{\gamma_{gj}}} - \phi_{gi} - \gamma_{gi} > \mu_g \frac{1}{2} - \phi_{gi} - \gamma_{gi}$$

. This is true if and only if $\frac{\gamma_{gj}}{\gamma_{gj} + (\gamma_{gi}+\gamma_{gj})^2} > \frac{1}{2}$. So notice that a sufficient condition is $\frac{\gamma_{gj}}{2} > (\gamma_{gi}+\gamma_{gj})^2$. Similar logic applies to $j$'s perspective.

Hence, if we have $(\gamma_{gi}+\gamma_{gj})^2 < \frac{1}{2}\min\{\gamma_{gi}, \gamma_{gj}\}$, both firms will not stop investing in Group $g$ even if the other were to unilaterally deviate to do so. Thus, this non-exiting condition will be satisfied for both firms if

$$(2\max_k \gamma_{gk})^2 \le \frac{1}{2}\min_k \gamma_{gk} \iff \min_k(\gamma_{gk}) > 8(\max_k \gamma_{gk})^2.$$

This condition is weaker than Inequality 9. In the case that $\gamma_{gi} = \gamma_{gj} = \gamma$, this is asking that $\frac{\gamma}{2} > (2\gamma)^2 \iff \gamma < \frac{1}{8}$.

Thus, if $\min_k \gamma_{gk} \ge (\max_k \gamma_{gk})^2$, then the interior equilibrium is the unique equilibrium.

□

### C.0.1 Technical Lemma for Simple Tullock Case.

We now supply the missing algebra from Lemma C.1:

Lemma C.1 (Technical Lemma). *The inequalities:*

$$(\gamma_{gi}+\gamma_{gj})^4 < \gamma^2_{gj}\left[\gamma_{gi} + (\gamma_{gi}+\gamma_{gj})^2\right],$$
$$(\gamma_{gi}+\gamma_{gj})^4 < \gamma^2_{gi}\left[\gamma_{gj} + (\gamma_{gi}+\gamma_{gj})^2\right]$$

*(inequalities 5 and 6) will be satisfied if* $\min_k \gamma_{gk} \ge 12(\max_k \gamma_{gk})^2$. *In the symmetric case, then* $\gamma_g \le \frac{1}{12}$.

Proof. The set we are interested in is the intersection of two solution sets to polynomial equations, and is hard to characterize precisely; however, we can give sufficient conditions on $\gamma_{gi}, \gamma_{gj}$ so that both inequalities are simultaneously satisfied.

We begin with the *symmetric* case, where $\gamma_{gi} = \gamma_{gj} = \gamma$, as it is easy to see: this is asking that

$$2^4\gamma^4 < \gamma^3 + 2^2\gamma^2\gamma^2 \iff 12\gamma^4 < \gamma^3 \iff \gamma < \frac{1}{12}. \tag{8}$$

If, instead, $\gamma_{gi} \ne \gamma_{gj}$, then we need to examine the algebra more carefully. We claim that the if $\max_k \gamma_k < \frac{1}{16}$ and $\min_k \gamma_{gk} > (\max_k \gamma_{gk})^2$ will suffice.

To see this, note that Inequality 5 expanded out is:

$$\begin{aligned} 0 &< \gamma_{gj}^2(\gamma_{gi} + (\gamma_{gj} + \gamma_{gi})^2) - (\gamma_{gi} + \gamma_{gj})^4 \\ &= \gamma_{gj}^2\gamma_{gi} + \gamma_{gj}^2(\gamma_{gi}^2 + \gamma_{gj}^2 + 2\gamma_{gi}\gamma_{gj}) \\ &\qquad - (\gamma_{gi}^4 + 4\gamma_{gi}^3\gamma_{gj} + 6\gamma_{gi}^2\gamma_{gj}^2 + 4\gamma_{gi}\gamma_{gj}^3 + \gamma_{gj}^4) \\ &= \gamma_{gj}^2\gamma_{gi} + \gamma_{gj}^2\gamma_{gi}^2 + \gamma_{gj}^4 + 2\gamma_{gi}\gamma_{gj}^3 \\ &\qquad - \gamma_{gi}^4 - 4\gamma_{gj}^3\gamma_{gj} - 6\gamma_{gi}^2\gamma_{gj}^2 - 4\gamma_{gi}\gamma_{gj}^3 - \gamma_{gj}^4 \\ &= \gamma_{gj}^2\gamma_{gi} - 5\gamma_{gj}^2\gamma_{gi}^2 - 2\gamma_{gi}\gamma_{gj}^3 - \gamma_{gi}^4 - 4\gamma_{gi}^3\gamma_{gj}. \end{aligned}$$

Now, notice that by replacing whichever of $\gamma_{gi}$ or $\gamma_{gj}$ with the larger of the two, we make the negative terms larger. So a sufficient (though again, not necessary) condition for the inequality to be satisfied is:

$$\gamma_{gj}^2\gamma_{gi} \geq 12(\max_k \gamma_{gk})^4,$$

But now notice that this is asking that either

$$(\max_k \gamma_{gk})^2 \min_k \gamma_{gk} \geq 12(\max_k \gamma_{gk})^4,$$

or

$$(\min_k \gamma_{gk})^2 \max_k \gamma_{gk} \geq 12(\max_k \gamma_{gk})^4, \tag{9}$$

depending on whether $\gamma_{gj} > \gamma_{gj}$ or vice versa. Since we can repeat the logic from equilibrium from Firm j's perspective, we will actually need *both* these conditions to hold for this point to be an equilibrium. But since $\max_{gk} \gamma_{gk} < 1$, it is sufficient that

$$\min_k \gamma_{gk} \geq 12(\max_k \gamma_{gk})^2,$$

which is simply asking that the firms are not too far apart in their marginal costs. □

*C.0.2 Proof of General Tullock case.* Our goal is to show the following:

LEMMA C.2. *Suppose two firms compete for proportional demand with parameters q and ρ. Suppose further that $\varepsilon_{gi}^* < 1$ for all g and i. If the* nondeviation condition *(as defined below) holds, then both firms playing $(\varepsilon_{gi}^*, \varepsilon_{gj}^*)$ is an equilibrium*

$$\varepsilon_{gi}^* = \left(\frac{q\gamma_{gi}}{\rho_g\mu_g}\right)^{\frac{1}{q}} \frac{(\gamma_{gi}^q + \gamma_{gj}^q)^{\frac{2}{q}}}{\gamma_{gi}^q\gamma_{gj}^q} = \left(\frac{q}{\rho_g\mu_g}\right)^{\frac{1}{q}} \frac{(\gamma_{gi}^q + \gamma_{gj}^q)^{\frac{2}{q}}}{\gamma_{gi}^{1-\frac{1}{q}}\gamma_{gj}^q}.$$

*If, furthermore, the* investment condition *(as defined below) holds, then this equilibrium is unique.*

PROOF. Under the proportional split model of demand, each firm's profit depends not only on its own action, but also that of the other firm. Again, this calls for a game theoretic notion of solution. We look for a pure strategy *Nash Equilibrium.* Recall that in an equilibrium, both firms must be best-responding and have no incentive to deviate.

To find an equilibrium, we first find the best-response of Firm $i$, *given* the choices of Firm $j$. Fixing $\varepsilon_j$, the profit of Firm $i$ given the choice of $\varepsilon$ is as follows:

$$\pi(\varepsilon_i, \varepsilon_j) = \sum_{g\in\mathcal{G}} \left[\mu_g \frac{\varepsilon_{gj}^{\rho_g}}{\varepsilon_{gi}^{\rho_g} + \varepsilon_{gj}^{\rho_g}}\right] - \sum_{g\in\mathcal{G}} \phi_{gi} + \frac{\gamma_{gi}}{\varepsilon_{gi}^q}.$$

Taking the derivative:

$$\frac{\partial\pi}{\partial\varepsilon_{gi}} = -\mu_g\varepsilon_{gj}^{\rho_g}(\varepsilon_{gi}^{\rho_g} + \varepsilon_{gj}^{\rho_g})^{-2}\left(\rho_g\varepsilon_{gi}^{\rho_g-1}\right) + \frac{q\gamma_{gi}}{\varepsilon_{gi}^{q+1}}.$$

Setting to zero yields the first-order condition:

$$\frac{q\gamma_{gi}}{\varepsilon_{gi}^{q+1}} = \frac{\rho_g\mu_g\varepsilon_{gi}^{\rho_g-1}\varepsilon_{gj}^{\rho_g}}{(\varepsilon_{gi}^{\rho_g} + \varepsilon_{gj}^{\rho_g})^2} \implies \frac{\rho_g\mu_g\varepsilon_{gi}^{\rho_g+q}\varepsilon_{gj}^{\rho_g}}{q\gamma_{gi}} = (\varepsilon_{gi}^{\rho_g} + \varepsilon_{gj}^{\rho_g})^2.$$

Applying symmetric logic to Firm $j$ and using the fact that the first order condition for each firm must hold simultaneously in equilibrium, we have that

$$\frac{\rho_g\mu_g\left(\varepsilon_{gi}^*\right)^{\rho_g+q}\left(\varepsilon_{gj}^*\right)^{\rho_g}}{q\gamma_{gi}} = \frac{\rho_g\mu_g\left(\varepsilon_{gj}^*\right)^{\rho_g+q}\left(\varepsilon_{gi}^*\right)^{\rho_g}}{q\gamma_{gj}}.$$

Solving for $\varepsilon_{gj}^*$ in terms of $\varepsilon_{gi}^*$ yields:

$$\varepsilon_{gj}^* = \varepsilon_{gi}^*\left(\frac{\gamma_{gj}}{\gamma_{gi}}\right)^{\frac{1}{q}}.$$

Substituting this back in, we have that

$$\begin{aligned} \frac{\rho_g\mu_g\left(\varepsilon_{gi}^*\right)^{\rho_g+q}\left(\varepsilon_{gi}^*\right)^{\rho_g}\left(\frac{\gamma_{gj}}{\gamma_{gi}}\right)^{\frac{\rho_g}{q}}}{q\gamma_{gi}} &= \left(\left(\varepsilon_{gi}^*\right)^{\rho_g} + \left(\varepsilon_{gi}^*\right)^{\rho_g}\left(\frac{\gamma_{gj}}{\gamma_{gi}}\right)^{\frac{\rho_g}{q}}\right)^2 \\ &= \left(\varepsilon_{gi}^*\right)^{2\rho}\left(1 + \left(\frac{\gamma_{gj}}{\gamma_{gi}}\right)^{\frac{\rho_g}{q}}\right)^2. \end{aligned}$$

Solving and rearranging gives that

$$\begin{aligned} \varepsilon_{gi}^* &= \left[\frac{q\gamma_{gi}}{\rho_g\mu_g}\left(\frac{\gamma_{gi}}{\gamma_{gj}}\right)^{\frac{\rho_g}{q}}\left(1 + \left(\frac{\gamma_{gj}}{\gamma_{gi}}\right)^{\frac{\rho_g}{q}}\right)^2\right]^{\frac{1}{q}} \\ &= \left[\frac{q\gamma_{gi}}{\rho_g\mu_g}\left(\frac{\gamma_{gi}}{\gamma_{gj}}\right)^{\frac{\rho_g}{q}}\left(\frac{\gamma_{gi}^{\frac{\rho_g}{q}} + \gamma_{gj}^{\frac{\rho_g}{q}}}{\gamma_{gi}^{\frac{\rho_g}{q}}}\right)^2\right]^{\frac{1}{q}} \\ &= \left[\frac{q\gamma_{gi}}{\rho_g\mu_g}\left(\frac{\gamma_{gi}}{\gamma_{gj}}\right)^{\frac{\rho_g}{q}}\frac{1}{\gamma_{gi}^{\frac{2\rho_g}{q}}}\left(\gamma_{gi}^{\frac{\rho_g}{q}} + \gamma_{gj}^{\frac{\rho_g}{q}}\right)^2\right]^{\frac{1}{q}} \\ &= \left[\frac{q}{\rho_g\mu_g}\frac{\left(\gamma_{gi}^{\frac{\rho_g}{q}} + \gamma_{gj}^{\frac{\rho_g}{q}}\right)^2}{\gamma_{gi}^{\frac{\rho_g}{q}}\gamma_{gj}^{\frac{\rho_g}{q}}}\right]^{\frac{1}{q}}. \end{aligned}$$

Now, notice that the profit can be written as

$$\pi_{gi}(\varepsilon^*_{gi}, \varepsilon^*_{gj}) = \frac{\left(\frac{q}{\rho_g\mu_g}\right)^{\frac{1}{q}} \frac{(\gamma^q_{gi}+\gamma^q_{gj})^{\frac{2}{q}}}{\gamma_{gi}^{1-\frac{1}{q}}\gamma^q_{gj}}}{\left(\frac{q}{\rho_g\mu_g}\right)^{\frac{1}{q}} \frac{(\gamma^q_{gi}+\gamma^q_{gj})^{\frac{2}{q}}}{\gamma_{gj}^{1-\frac{1}{q}}\gamma^q_{gi}} + \left(\frac{q}{\rho_g\mu_g}\right)^{\frac{1}{q}} \frac{(\gamma^q_{gi}+\gamma^q_{gj})^{\frac{2}{q}}}{\gamma_{gi}^{1-\frac{1}{q}}\gamma^q_{gj}}} - \frac{\gamma_{gi}}{\left(\varepsilon^*_{gi}\right)^q}$$

$$= \frac{\left(\frac{1}{\gamma_{gi}^{1-\frac{\rho_g}{q}}\gamma^q_{gj}}\right)^{\rho}}{\left(\frac{1}{\gamma_{gi}^{1-\frac{\rho_g}{q}}\gamma^q_{gj}}\right)^{\rho} + \left(\frac{1}{\gamma_{gj}^{1-\frac{\rho_g}{q}}\gamma^q_{gi}}\right)^{\rho}} - \frac{\gamma_{gi}}{\left(\varepsilon^*_{gi}\right)^q}$$

$$= \frac{1}{1+\left[\frac{\gamma_{gj}^{1-\rho_g/q}\gamma^q_{gi}}{\gamma_{gi}^{1-\rho_g/q}\gamma^q_{gj}}\right]^{\rho_g}} - \frac{\gamma_{gi}}{\left(\varepsilon^*_{gi}\right)^q}$$

$$= \frac{1}{1+\left[\frac{\gamma_{gj}^{1-\rho_g/q-q}}{\gamma_{gi}^{1-\rho_g/q-q}}\right]^{\rho_g}} - \frac{\gamma_{gi}}{\left(\varepsilon^*_{gi}\right)^q}.$$

Substituting back in $\left(\varepsilon^*_{gi}\right)^q$, it is:

$$\frac{1}{1+\left[\frac{\gamma_{gj}^{1-\rho_g/q-q}}{\gamma_{gi}^{1-\rho_g/q-q}}\right]^{\rho_g}} - \frac{\gamma_{gi}\rho_g\mu_g\gamma_{gi}^{\frac{\rho_g}{q}}\gamma_{gj}^{\frac{\rho_g}{q}}}{q\left(\gamma_{gi}^{\frac{\rho_g}{q}}+\gamma_{gj}^{\frac{\rho_g}{q}}\right)^2}.$$

For this interior equilibrium to hold, it must be that $\pi^*_{gi}(\varepsilon^*_{gi}, \varepsilon^*_{gj}) \geq \pi^*_{gi}(\varepsilon', \varepsilon^*_{gj})$ for all other choices $\varepsilon'$. Note that $\pi_{gi,\varepsilon^*_{gj}}(\varepsilon)$ is continuous away from 0. Moreover, for small enough $\varepsilon$, $\pi_{gi,\varepsilon^*_{gj}}(\varepsilon) < 0$, since the market size is bounded by costs can be come arbitrarily negative. Hence, we can consider maximizing this function on the compact set $[\varepsilon_0, 1]$, where $\varepsilon_0$ is the point at which profit becomes negative. Since $\pi_{gi,\varepsilon^*_{gj}}(\varepsilon)$ is continuous on this set, and $\varepsilon^*_{gi}$ satisfies the first-order condition, the only possible maxima of this function are $\varepsilon_0$ or 1. At $\varepsilon_0$, the firm is making zero profits, so any choice with positive profits eliminates it. At $\varepsilon = 1$, the firm can also choose to not invest anything in data (and receive the same revenue but no data costs), so the condition that makes $\pi_{gi,\varepsilon^*_{gj}}(\varepsilon^*_{gi}) > \pi_{gi,\varepsilon^*_{gj}}(1)$ will be sufficient to make this an equilibrium.

This condition holds if

$$\frac{1}{1+\left[\frac{\gamma_{gj}^{1-\rho_g/q-q}}{\gamma_{gi}^{1-\rho_g/q-q}}\right]^{\rho_g}} - \frac{\gamma_{gi}\rho_g\mu_g\gamma_{gi}^{\frac{\rho_g}{q}}\gamma_{gj}^{\frac{\rho_g}{q}}}{q\left(\gamma_{gi}^{\frac{\rho_g}{q}}+\gamma_{gj}^{\frac{\rho_g}{q}}\right)^2} \geq \pi_{gi}(1, \varepsilon_{gj^*}). \qquad (10)$$

We call Inequality 10 the *nondeviation condition*. We can write:

$$\pi_{gi}(1, \varepsilon_{gj^*}) = \frac{\left(\frac{q}{\rho_g\mu_g}\right)^{\frac{1}{q}} \frac{(\gamma^q_{gi}+\gamma^q_{gj})^{\frac{2}{q}}}{\gamma_{gi}^{1-\frac{1}{q}}\gamma^q_{gj}}}{1+\left(\frac{q}{\rho_g\mu_g}\right)^{\frac{1}{q}} \frac{(\gamma^q_{gi}+\gamma^q_{gj})^{\frac{2}{q}}}{\gamma_{gi}^{1-\frac{1}{q}}\gamma^q_{gj}}}$$

$$= \frac{1}{1+\left(\frac{q}{\rho_g\mu_g}\right)^{-\frac{1}{q}} \frac{(\gamma^q_{gi}+\gamma^q_{gj})^{\frac{-2}{q}}}{\gamma_{gi}^{\frac{1}{q}-1}\gamma^{-q}_{gj}}},$$

so Inequality 10 asks that

$$\frac{1}{1+\left[\frac{\gamma_{gj}^{1-\rho_g/q-q}}{\gamma_{gi}^{1-\rho_g/q-q}}\right]^{\rho_g}} - \frac{\gamma_{gi}\rho_g\mu_g\gamma_{gi}^{\frac{\rho_g}{q}}\gamma_{gj}^{\frac{\rho_g}{q}}}{q\left(\gamma_{gi}^{\frac{\rho_g}{q}}+\gamma_{gj}^{\frac{\rho_g}{q}}\right)^2} \geq \frac{1}{1+\left(\frac{q}{\rho_g\mu_g}\right)^{-\frac{1}{q}} \frac{(\gamma^q_{gi}+\gamma^q_{gj})^{\frac{-2}{q}}}{\gamma_{gi}^{\frac{1}{q}-1}\gamma^{-q}_{gj}}}.$$

We have shown that if the nondeviation condition holds for each group and each firm, then $(\varepsilon^*_{gi}, \varepsilon^*_{gj})$ is a Nash Equilibrium in pure strategies under proportionally split demand with competition exponent $\rho_g$ in each group and learning rate $q$. If a further condition holds, namely that there exists a preferred strategy to non-investment if the opponent invests, then the equilibrium is unique.

Call this the *investment* condition: there exists $\varepsilon \in (0, 1)$ such that:

$$\frac{1}{\varepsilon_g^{\rho}+1} - \frac{\gamma_{gi}}{\varepsilon^q} > \frac{\mu_g}{2} \iff \varepsilon^q - \gamma_{gi}(\varepsilon^{\rho_g}+1) \geq \frac{\mu_g\left((\varepsilon^{\rho_g}+1)\varepsilon^q\right)}{2}. \qquad (11)$$

Equivalently, we need to ensure that there is an $\varepsilon \in (0, 1)$ such that:

$$\iff \varepsilon^q - \gamma_{gi}(\varepsilon^{\rho_g}+1) - \frac{\mu_g\left((\varepsilon^{\rho_g}+1)\varepsilon^q\right)}{2} \geq 0 \qquad (12)$$

has a solution in $(0, 1)$. This will not always be the case, of course; when it is not, then there is an equilibrium in which both firms prefer not to invest in collecting data from one group at all, which certainly exacerbates inequality. □

# D OMITTED PROOFS FROM SECTION 6.1

Omitted Algebra for Optimal Profit. Recall that we would like to show that the optimal profit achievable by the monopolist facing parameters $\mu_g, \alpha_g, \beta_g, \gamma_g$ for $g$ in $\mathcal{G} = \{A, B\}$ is:

$$\pi^*(\varepsilon^*) = \sum_{g\in\{A,B\}} \mu_g\alpha_g - (\mu_g\beta_g)^{\frac{q}{q+1}}\gamma_g^{\frac{1}{q+1}}Q.$$

For clarity, write $\eta_g$ for $\mu_g\beta_g$. Then we can write the optimal profit for a group as a function of the parameters using the result that the profit optimizing choice of error is $\varepsilon_g^* = (q\gamma_g/\eta_g)^{1/(q+1)}$.

$$\begin{aligned}\pi_g^*(\mu_g, \gamma_g, \alpha_g, \beta_g) &= \alpha_g\mu_g - \eta_g\varepsilon_g^* - \gamma/{\varepsilon_g^*}^q \\ &= \alpha_g\mu_g - \eta_g\eta_g^{-\frac{1}{q+1}} q^{\frac{1}{q+1}} \gamma_g^{\frac{1}{q+1}} - \gamma_g\left(\frac{\eta_g}{q\gamma_g}\right)^{\frac{q}{q+1}} \\ &= \alpha_g\mu_g - \eta_g^{\frac{q}{q+1}}\gamma_g^{\frac{1}{q+1}} q^{\frac{1}{q+1}} - \gamma_g^{\frac{1}{q+1}}\frac{1}{q^{q/(q+1)}}\eta_g^{\frac{q}{q+1}} \\ &= \alpha_g\mu_g - \eta_g^{\frac{q}{q+1}}\gamma_g^{\frac{1}{q+1}}\left[q^{\frac{1}{q+1}} + \frac{1}{q^{q/(q+1)}}\right].\end{aligned}$$

Then writing $Q = q^{\frac{1}{q+1}} + \frac{1}{q^{q/(q+1)}}$, substituting back $\mu_g\beta_g$ for $\eta_g$, and summing over groups yields the claim. □

Proof of Lemma 6.3. Fix a solution $(\varepsilon_A, \varepsilon_B)$ to the constrained profit optimization problem. We will show that unless $\varepsilon_B = (1+\chi)\varepsilon_A$, $(\varepsilon_A, \varepsilon_B)$ is not a constrained profit maximizer.

Since by assumption $\varepsilon_B^M > \varepsilon_A^M(1+\chi)$ but $\varepsilon_A/(1+\chi) \le \varepsilon_B \le (1+\chi)\varepsilon_A$, we can't have both $\varepsilon_B^M = \varepsilon_B$ and $\varepsilon_A^M = \varepsilon_A$. Without loss of generality, assume that $\varepsilon_B^M \ne \varepsilon_B$.

There are three cases. In the first case, $\varepsilon_B^M > (1+\chi)\varepsilon_A$. We can increase the profit achieved by $(\varepsilon_A, \varepsilon_B)$ by increasing $\varepsilon_B$, as in this case, $\varepsilon_B < \varepsilon_B^M$. To see this, let $\varepsilon_\alpha = \left(\varepsilon_A, \alpha\varepsilon_B^M + (1-\alpha)\varepsilon_B\right)$ for $\alpha \in [0,1]$. By Jensen's inequality, there is an $\alpha$ such that

$$\pi(\varepsilon_\alpha) \ge (1-\alpha)\pi\left((\varepsilon_A, \varepsilon_B)\right) + \alpha\pi\left(\left(\varepsilon_A, \varepsilon_B^M\right)\right) > \pi((\varepsilon_A, \varepsilon_B)).$$

The first inequality holds for any $\alpha \in [0,1]$, so we set $\alpha$ so that $\varepsilon_\alpha = (\varepsilon_A, (1+\chi)\varepsilon_A)$, in which case this is still a feasible solution, and by the separability of the profit function, the second inequality holds.

In the second case, $\varepsilon_B^M < \varepsilon_A/(1+\chi)$. Then by the same logic using Jensen's inequality, we can increase the profit by decreasing $\varepsilon_B$ to $\varepsilon_A/(1+\chi)$, i.e. $\pi((\varepsilon_A, \varepsilon_B)) < \pi((\varepsilon_A, \varepsilon_A/(1+\chi)))$. But we can increase the profit even more in this case because $\varepsilon_A^M < \varepsilon_B^M/(1+\chi) < \varepsilon_A/(1+\chi)^2$, so now we can decrease $\varepsilon_A$ to see that profit is maximized at $\left(\varepsilon_A/(1+\chi)^2, \varepsilon_A/(1+\chi)\right)$.

Otherwise, $\varepsilon_A/(1+\chi) \le \varepsilon_B^M \le (1+\chi)\varepsilon_A$. This is very similar to the previous case: Jensen's inequality along with the separability of $\pi$ shows that $\pi((\varepsilon_A, \varepsilon_B)) < \pi\left(\left(\varepsilon_A, \varepsilon_B^M\right)\right) \le \left(\varepsilon_B^M/(1+\chi), \varepsilon_B^M\right)$. □

Proof of Corollary 6.5. First, we show that $\varepsilon_A^R \ge \varepsilon_A^M$:

$$\begin{aligned}\left(\frac{\varepsilon_A^R}{\varepsilon_A^M}\right)^{q+1} &= \frac{q\left(\gamma_A + \gamma_B/(1+\chi)^q\right)/(\mu_A\beta_A + \mu_B\beta_B(1+\chi))}{q\gamma_A/(\mu_A\beta_A)} \\ &= \frac{\mu_A\beta_A}{\mu_A\beta_A + \mu_B\beta_B(1+\chi)}\frac{\gamma_A + \gamma_B/(1+\chi)^q}{\gamma_A} \\ &= \frac{\mu_A\beta_A\gamma_A + \mu_A\beta_A\gamma_B/(1+\chi)^q}{\mu_A\beta_A\gamma_A + \mu_B\beta_B\gamma_A(1+\chi)}.\end{aligned}$$

Notice that

$$\frac{\varepsilon_A^R}{\varepsilon_A^M} \ge 1 \iff \left(\frac{\varepsilon_A^R}{\varepsilon_A^M}\right)^{q+1} \ge 1.$$

Now using the elementary fact that for positive $x, y, z$, $(x+y)/(x+z) \ge 1 \iff y \ge z$, we can see that

$$\begin{aligned}\left(\frac{\varepsilon_A^R}{\varepsilon_A^M}\right)^{q+1} \ge 1 &\iff \mu_A\beta_A\gamma_B/(1+\chi)^q \ge \mu_B\beta_B\gamma_A(1+\chi) \\ &\iff \frac{\gamma_B}{\mu_B\beta_B} \ge \frac{\gamma_A}{\mu_A\beta_A}(1+\chi)^{q+1}.\end{aligned}$$

But recalling that the monopolist's optimal solution is $\varepsilon_g^M = (\frac{q\gamma_g}{\mu_g\beta_g})^{\frac{1}{q+1}}$, we can rewrite the previous inequality as

$${\varepsilon_B^M}^{q+1} \ge {\varepsilon_A^M}^{q+1}(1+\chi)^{q+1} \iff \varepsilon_B^M \ge \varepsilon_A^M(1+\chi),$$

which is exactly Assumption 6.1.

Now, we show that $\varepsilon_B^R \le \varepsilon_B^M$:

$$\varepsilon_B^R = (1+\chi)\varepsilon_A^R = \left(q\frac{((1+\chi)^{q+1}\gamma_A + \gamma_B(1+\chi))}{\mu_A\beta_A + \mu_B\beta_B(1+\chi)}\right)^{\frac{1}{q+1}}.$$

Then

$$\begin{aligned}\left(\frac{\varepsilon_B^R}{\varepsilon_B^M}\right)^{q+1} &= q\frac{\gamma_A(1+\chi)^{q+1} + \gamma_B(1+\chi)}{\mu_A\beta_A + \mu_B\beta_B(1+\chi)}\frac{\mu_B\beta_B}{q\gamma_B} \\ &= \frac{\mu_B\beta_B\gamma_A(1+\chi)^{q+1} + \mu_B\beta_B(1+\chi)\gamma_B}{\gamma_A\mu_A\beta_A + \mu_B\beta_B\gamma_B(1+\chi)}.\end{aligned}$$

So again using the elementary fact that $\frac{y+x}{z+x} \iff y < z$, we have:

$$\begin{aligned}\left(\frac{\varepsilon_B^R}{\varepsilon_B^M}\right)^{q+1} \le 1 &\iff \mu_B\beta_B\gamma_B(1+\chi) \le \mu_A\beta_A\gamma_A \\ &\iff \frac{\gamma_B}{\mu_B\gamma_B} \ge \frac{\gamma_A}{\mu_A\beta_A}(1+\chi)^{q+1} \\ &\iff {\varepsilon_B^M}^{q+1} \ge {\varepsilon_A^M}^{q+1}(1+\chi)^{q+1} \\ &\iff \varepsilon_B^M \ge \varepsilon_A^M(1+\chi).\end{aligned}$$

□

Proof of Corollary 6.6. Returning to the second equation in the proof of Corollary 6.5 :

$$\begin{aligned}\left(\mathrm{PoF}_{1+\chi}\right)^{q+1} &= \frac{\mu_A\beta_A}{\mu_A\beta_A + \mu_B\beta_B(1+\chi)}\frac{\gamma_A + \gamma_B/(1+\chi)^q}{\gamma_A} \\ &\le \frac{\gamma_A + \gamma_B/(1+\chi)^q}{\gamma_A}.\end{aligned}$$

Taking the $(q+1)$th-root yields the claim. □

Proposition 6.9. We can write

$$\mathrm{MonPoF}_{1+\chi} = \frac{\mu_A\alpha_A + r\mu_A\alpha_B - Q\mu_A^{\frac{q}{q+1}}\left[\beta_A^{\frac{q}{q+1}}\gamma_A^{\frac{1}{q+1}} + r^{\frac{q}{q+1}}\beta_B^{\frac{q}{q+1}}\gamma_B^{\frac{1}{q+1}}\right]}{\mu_A\alpha_A + r\mu_A\alpha_B - Q\mu_A^{\frac{q}{q+1}}\left[(\beta_A + r\beta_B(1+\chi))^{\frac{q}{q+1}}(\gamma_A + \gamma_B\frac{1}{(1+\chi)}^q)^{\frac{1}{q+1}}\right]}$$

Factoring out $\mu_A$ and using the fact that if $\mu_B \to \infty$, $\mu_A \to \infty$, we have:

$$\begin{aligned}\lim_{\mu_B\to\infty}\mathrm{MonPoF}_{1+\chi} &= \lim_{\mu_A\to\infty}\mathrm{MonPoF}_{1+\chi} \\ &= \lim_{\mu_A\to\infty}\frac{\alpha_A + r\alpha_B - Q\mu_A^{-\frac{1}{q+1}}\left[\beta_A^{\frac{q}{q+1}}\gamma_A^{\frac{1}{q+1}} + r^{\frac{q}{q+1}}\beta_B^{\frac{q}{q+1}}\gamma_B^{\frac{1}{q+1}}\right]}{\alpha_A + r\alpha_B - Q\mu_A^{-\frac{1}{q+1}}\left[(\beta_A + r\beta_B(1+\chi))^{\frac{q}{q+1}}(\gamma_A + \gamma_B\frac{1}{(1+\chi)}^q)^{\frac{1}{q+1}}\right]}\end{aligned}$$

But then

$$\lim_{\mu_B \to \infty} \text{MonPoF}_{1+\chi} = 1,$$

since $\mu_A^{-1/(q+1)} \to 0$ as $\mu_A \to \infty$ and its multipliers are constants.

For the second claim, we can simply substitute in $\mu_B = 0$ and factor out $\mu_A \alpha_A$ to get

$$\begin{aligned}\lim_{\mu_B \to 0} \text{MonPoF} &= \frac{\mu_A \alpha_A [1 - Q\left(\frac{\gamma_A}{\mu_A \alpha_A}\right)^{\frac{1}{q+1}}]}{\mu_A \alpha_A [1 - Q\left(\frac{\gamma_A + \gamma_B/(1+\chi)^{\frac{1}{q+1}}}{\mu_A \alpha_A}\right)^{\frac{1}{q+1}}]} \\ &= \frac{[1 - Q\left(\frac{\gamma_A}{\mu_A \alpha_A}\right)^{\frac{1}{q+1}}]}{1 - Q\left[\frac{\gamma_A + \gamma_B/(1+\chi)^{\frac{1}{q+1}}}{\mu_A \alpha_A}\right]^{\frac{1}{q+1}}} \\ &= \frac{1 - Q/q\varepsilon_A^M}{1 - Q/q\varepsilon_A^M \left(1 + \frac{\gamma_B}{\gamma_A} \frac{1}{(1+\chi)^{\frac{1}{q+1}}}\right)^{\frac{1}{q+1}}}.\end{aligned}$$

□

## E OMITTED PROOFS FROM SECTION 6.2

Proof of Lemma 6.11. First, note that the absolute error constraints are separable, so that the firm's profit maximization problem is simply $\max_{\varepsilon_g} \pi_g(\varepsilon_g)$ subject to $\varepsilon_g \leq \chi$, for each of $g \in \{A, B\}$. So fix a group $g \in \{A, B\}$. If $\varepsilon_g^R \neq \varepsilon_g^M$, it must be that $\varepsilon_g^M > \chi$, as otherwise the constraint would have already been met by $\varepsilon_g^M$. Now we show that for any feasible error rate $\varepsilon_g \leq \chi$, $\pi_g(\varepsilon_g) \leq \pi_g(\chi)$ with equality holding only at $\varepsilon_g = \chi$. So suppose that $\varepsilon_g < \chi$.

Let $\varepsilon_{g,\alpha} = (1-\alpha)\varepsilon_g + \alpha\varepsilon_g^M$ for $\alpha \in [0,1]$. Since $\pi_g$ is concave, Jensen's inequality implies that

$$\begin{aligned}\pi_g(\varepsilon_{g,\alpha}) &= \pi_g\left((1-\alpha)\varepsilon_g + \alpha\varepsilon_g^M\right) \\ &\geq (1-\alpha)\pi_g(\varepsilon_g) + \alpha\pi_g\left(\varepsilon_g^M\right) \\ &> \pi_g(\varepsilon_g),\end{aligned}$$

with the second inequality holding when $\alpha > 0$ because $\pi_g\left(\varepsilon_g^M\right) > \pi_g(\varepsilon_g)$. Then choosing $\alpha = \frac{\chi - \varepsilon_g}{\varepsilon_g^M - \varepsilon_g}$ suffices as then $\varepsilon_{g,\alpha} = \chi$. (Notice that $\frac{\chi - \varepsilon_g}{\varepsilon_g^M - \varepsilon_g} \in (0,1]$ since $\varepsilon_g^M > \chi$ and $\chi > \varepsilon_g$.)

□

Proof of Proposition 6.12. First, by Lemma 6.11, when $\chi < \varepsilon_A^M$, then the optimal choice for the monopolist is $(\varepsilon_A^R, \varepsilon_B^R) = (\chi, \chi)$. In this case, the profit is

$$\pi(\chi,\chi) = \mu_A(\alpha_A - \beta_A\chi) + \mu_B(\alpha_B - \beta_B\chi) - \phi_A - \phi_B - \frac{\gamma_A}{\chi^q} - \frac{\gamma_B}{\chi^q}.$$

Since $\pi$ is concave, the global optimum is at $(\varepsilon_A^M, \varepsilon_B^M)$, and as the error rate goes to zero, profit goes to negative infinity, there must be a minimum error rate $\chi_0 > 0$ where the profit is zero. Since this error rate must be smaller than $\varepsilon_A^M$, the above formula for profit holds and this error rate is the solution to

$$\mu_A(\alpha_A - \beta_A\chi) + \mu_B(\alpha_B - \beta_B\chi) - \phi_A - \phi_B - \frac{\gamma_A}{\chi^q} - \frac{\gamma_B}{\chi^q} = 0.$$

Multiplying by $\chi^q$ and re-arranging gives the claim. □

Proof of Corollary 6.15. For $\mu_B \to \infty$, note that we can write the price of fairness as:

$$\begin{cases} 1 & \chi \geq \varepsilon_B^M \\ \frac{\pi_A(\varepsilon_A^M)+\pi_B(\varepsilon_B^M)}{\pi_A(\varepsilon_A^M)+\pi_B(\chi)} & \varepsilon_A^M \leq \chi < \varepsilon_B^M \\ \frac{\pi_A(\varepsilon_A^M)+\pi_B(\varepsilon_B^M)}{\pi_A(\chi)+\pi_B(\chi)} & \chi_0 < \chi < \varepsilon_A^M \\ \infty & \chi < \chi_0 \end{cases}.$$

Since $\varepsilon_A^M, \varepsilon_B^M \to 0$ as $\mu_B, \mu_A \to \infty$, as the population grows, eventually $\varepsilon_B^M$ and $\varepsilon_A^M$ will be less than $\chi$, so that $\varepsilon_B^R = \varepsilon_B^M$ and $\varepsilon_A^R = \varepsilon_A^M$. Thus $\lim_{\mu_B \to \infty} \text{MonPoF}_\chi = 1$.

For $\mu_B \to 0$, given our assumption on $\chi$, we will be be in either Case 1, Case 2, or Case 3. Note that as $\mu_B \to 0$, $\varepsilon_B^M$ will eventually be larger than $\chi$, so the limit will be obtained at either Case 2 or Case 3. In Case 2, we can substitute in 0 for $\mu_B$; combining this with the fact that for small enough $\mu_B$, the optimal choice for the unconstrained monopolist eventually becomes to set $\varepsilon_B^M = 1$, we can write the price of fairness as:

$$\begin{aligned}\lim_{\mu_B \to \infty} \text{MonPoF}_\chi &= \frac{\pi_A(\varepsilon_A^M) - \gamma_B - \phi_B}{\pi_A(\varepsilon_A)^M - \gamma_B/\chi^q - \phi_B} \geq 1 \\ &= \frac{\mu_A\alpha_A - Q\gamma_A^{\frac{1}{q+1}}(\mu_A\beta_A)^{\frac{q}{q+1}} - \phi_A - \gamma_B - \phi_B}{\mu_A\alpha_A - Q\gamma_A^{\frac{1}{q+1}}(\mu_A\beta_A)^{\frac{q}{q+1}} - \phi_A - \gamma_B/\chi^q - \phi_B} \\ &= \frac{1 - \frac{\gamma_B - \phi_B}{\frac{1}{q+1}(\mu_A\beta_A)^{\frac{q}{q+1}} - \phi_A}}{1 - \frac{\gamma_B/\chi^q - \phi_B}{\frac{1}{q+1}(\mu_A\beta_A)^{\frac{q}{q+1}} - \phi_A}} \geq 1.\end{aligned}$$

since $\chi \leq 1 \implies \gamma_B/\chi^q \geq \gamma_B$.

Alternatively, if Case 3 obtains, then we can write:

$$\lim_{\mu_B \to \infty} \text{MonPoF}_\chi = \frac{\pi_A(\chi) - \gamma_B - \phi_B}{\pi_A(\chi) - \gamma_B/\chi^q - \phi_B} \geq 1.$$

□